

# Discrete ordered $p$ -median problems with neighbourhoods

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**ABSTRACT.** In this paper we introduce a new variant for the  $p$ -median facility location problem in which it is assumed that the exact position of the potential facilities is unknown. Instead, each of the facilities must belong to a convex region around their initial estimated positions. In this problem, two main decisions have to be made simultaneously: the determination of the potential facilities that must be opened to serve the demands of the customers and the location of the open facilities into their neighborhoods, at global minimum cost. We present several MINLP formulations for a wide family of objective functions which are usual in Location Analysis: ordered median functions. We also describe two math-heuristic approaches for solving the problem. The results of an extensive computational experience are reported.

## 1. INTRODUCTION

Facility location finds the optimal placement of one or several new facilities/plants among the potential sites to satisfy the demand of a set of customers. In discrete facility location the position of both the customers and the potential new facilities are part of the input information as well as the travel costs between each of them. On the other hand, in continuous facility location problems, although the (geographical) coordinates (in a  $d$ -dimensional space) of each the customers are provided, the information about the location of the facilities is inexact, in the sense that the facilities can be located at any place in the given space. Both the discrete and the continuous version of facility location problems have been widely studied in the literature (see the monographs [16, 32] and the references therein). Several versions of both discrete and continuous location problems have been analyzed by considering different objective functions [44], by fixing either the number of facilities to be located (as in the  $p$ -median or  $p$ -center problems) [19] or maximum capacities for the facilities (capacitated facility location)[24, 37], by assuming uncertainty into the demands of the clients (see [2, 11, 10] for a recent review), or combined with routing problems [8], amongst many others.

In this paper we study a new unified framework for discrete facility location problems in which the underlined problem is a discrete facility location problem, but because of imprecision, lack of information, or some kind of locational uncertainty, the new facilities are allowed to be located not only in the exact point of potential facilities but in certain regions around each of them, the *neighborhoods*. In case the facilities placements are precise enough, the neighborhoods can be assumed to be singletons (with unique elements the initial sites for the facilities), and the problem becomes the underlined version of the problem. On the other hand, if the neighborhoods are large enough, the problem turns into the continuous location version of the problem, allowing the facilities to be located in the whole space. In

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other cases, different neighborhoods shapes and sizes allows to model how imprecise is the provided locational information. The goal is, apart from the decision of the discrete location problem (placement of facilities and allocations customers-plants), to find the optimal location of the opened facilities inside the neighborhoods. The main difference between this problem and its underlined discrete facility location problem, is that in the latest the travel distances between facilities and clients are assumed to be known, while in the neighborhood version of the problem, those distances depend of the place where the facility is located in the neighborhood. Hence, in this new problem a matrix of travel costs is not provided, but a distance measure to compute the travel costs is assumed to be given. This problem, as far as we know, has not been previously investigated in location theory, although some attempts were presented in [9] and [20] where sensitivity analyses were performed by allowing the customers to *move* around disk neighborhoods on the plane. Also, this variant can be seen as a constrained version of the classical multifacility location problem, which have been only partially studied in the literature [6]. The new framework will be called *Discrete Facility Location with Neighborhoods*, terminology borrowed from the neighborhood versions of Minimum Spanning Trees [4, 13, 15, 47] and Traveling Salesman problems [17, 12].

The importance of analysing this family of problems comes from the wide range of applicability in practice. It is well known that discrete facility location problems are useful in many real-world applications (see [31, 26], amongst many others). However, in many situations, as for instance in the design of telecommunication networks, where a set of servers must be located to supply connection to a set of clients, the exact location of a server may not be exactly provided, but a region where it should be located (as in a corridor, a room, or any other bounded space) can be *easily* determined by the decision maker. In such a case, a robust worst-case decision is not desirable since the decision maker does not know the location of the facility not because a matter of uncertainty but because locational flexibility in the decision. An optimal design can be obtained if those neighborhoods are allowed for the facilities location.

In this paper, we provide mathematical programming formulations for the neighborhood versions of a widely studied family of discrete facility location problems: discrete ordered  $p$ -median problems (DOMP). In DOMP,  $p$  facilities must be located by minimizing a flexible objective function that allows to model different classical location problems. For instance, ordered median problems allows to model location problems in which the customers support the median ( $p$ -median) or the maximum ( $p$ -center) travel costs, among many other robust alternatives. Ordered median problems were introduced in Location Analysis by Puerto and Fernández [38] and several papers have analyzed this family of problems, not only for discrete location problems [22, 25, 36], but also in continuous location [3, 5], in network/tree location [21, 43] or hub location [40] (see [42] for a recent overview on the development of ordered median location problems). In particular, we will analyze the neighborhood version of the discrete ordered  $p$ -median problem for the so-called monotone case. In order to be able efficiently handle neighborhoods, we study the case in which they are defined second-order cone representable sets. These sets are able to model, as particular cases balls of  $\ell_\tau$  norms, for  $\tau \geq 1$ . The distances measures to represent travel costs between customers and facilities are assumed to be  $\ell_\nu$ -norm based distances for any  $\nu \geq 1$ . Within this framework we state four different Mixed Integer Second Order Cone Programming formulations. In order to solve larger instances, we propose two math-heuristic local search-based algorithms. The current limitations of the on-the-shelf solvers to solve mixed integer nonlinear problems and

the difficulty of solving even the underlined problem (DOMP is NP-hard), makes the resolution of the DOMP a challenge.

Our paper is organized in five sections. The first is our introduction. In Section 2 we introduce the framework and some general properties are studied. Section 3 is devoted to provide several mixed integer non linear programming formulations of the problem. We run some computational experiments in order to compare the performance of each formulation. In Section 4 two local search based math-heuristics are described, and the results of some computational experiments are reported. Finally, some conclusions and future research line on the problem are presented in Section 5.

## 2. DOMP WITH NEIGHBORHOODS

In this section we introduce the Discrete Ordered Median Problem with Neighbourhoods (DOMP) in which the underlined discrete facility location problem is the Discrete Ordered  $p$ -Median Problem (DOMP).

For the sake of presentation, we first describe the DOMP problem. Let  $\mathcal{A} = \{a_1, \dots, a_n\}$  be the set of coordinates of the customers in  $\mathbb{R}^d$ . We also assume, as usual in the location literature, that the coordinates of potential facilities coincides with  $\mathcal{A}$ , i.e., any customer is a potential facility. A matrix of travel costs between facilities is also provided  $D = (d(a_i, a_j))_{i,j=1}^n$ . A set of weights  $\lambda_1, \dots, \lambda_n \geq 0$  is also given in order to define the objective function of the problem. Set-up costs for each facility are considered:  $f(a) \in \mathbb{R}_+$ , for each  $a \in \mathcal{A}$ .

The goal of DOMP is to select, from the elements of  $\mathcal{A}$ , a subset of  $p$  facilities,  $\mathcal{B} \subset \mathcal{A}$  with  $|\mathcal{B}| = p$ , that minimizes the ordered median objective function  $\sum_{i=1}^n \lambda_i D_{(i)}$ , where  $D_i = \min_{b \in \mathcal{B}} d(a_i, b)$  (the distance from customer  $i$  to its closest open facility), and  $D_{(i)}$  represent the  $i$ -th largest element in the set  $\{D_1, \dots, D_n\}$ . Thus, the DOMP can be stated as the following optimization problem:

$$(\text{DOMP}) \quad \min_{\mathcal{B} \subset \mathcal{A}: |\mathcal{B}|=p} \sum_{i=1}^n \lambda_i D_{(i)} + \sum_{b \in \mathcal{B}} f(b)$$

We will assume that the  $\lambda$ -weights verify  $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ , dealing with the so-called *convex ordered median problem*. Most of the main well-known objective functions in Locational Analysis are part of the convex ordered median functions family are shown in Table 1.

	$\lambda$	Ordered Median Function
Median	$(1, \dots, 1)$	$\sum_{i=1}^n D_i$
Center	$(1, 0, \dots, 0)$	$\max_{i=1, \dots, n} D_i$
$K$ -Centrum	$(\overbrace{1, \dots, 1}^K, 0, \dots, 0)$	$\sum_{i=1}^K D_{(i)}$
Cent-Dian $_{\alpha}$ ( $\alpha \leq 1$ )	$(1, 1 - \alpha, \dots, 1 - \alpha)$	$\alpha \max_{i=1, \dots, n} D_i + (1 - \alpha) \sum_{1 \leq i \leq n} D_i$

TABLE 1. Some classical convex ordered median problems.

The reader is referred to [42] for a complete description of the properties of ordered median function. In general, ordered median function are continuous, symmetric, and since we assume that  $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ , the ordered median function is also convex, fact that will be exploited throughout this paper.

Some formulations and solution approaches for DOMP have appeared in the literature since the problem was introduced. In particular in Boland et. al [7] the problem is formulated as a (non convex) quadratic problem with quadratic constraints. A three index (pure) binary programming reformulation with  $O(n^3)$  variables and  $O(n^2)$  constraints was provided by linearizing the bilinear terms. A second formulation, reducing to two the indices of the variables in the formulation was also presented in the same paper by using a different linearization of their original formulation, reducing to  $O(n^2)$  the number of variables in the model. Puerto [37], Marin et. al [29], Marin et. al [28] and Labbé et. al [25] provided different formulations with three or two indices which need, in preprocessing phase, sorting the given travel distances (and removing duplicates). All the mentioned formulations are valid for the general case, not only for the convex version of the problem. Ogryzack and Tamir [35] presented a different formulation which exploits the convexity of the monotone  $\lambda$ -weights case by inducing a  $k$ -sums representation of the ordered median function (see also the recent paper [41] for further details on the powerful of this representation in a wide variety of optimization problem). Finally, in Blanco et. al [5] the authors derive a formulation that also avoid using the binary variables for sorting the involved distances, which uses the assignments constraints identifying permutations of the indices when sorting the distances. Some of this formulations extend to the DOMP. Also, a few heuristic approaches are available in the literature for the DOMP problem (see [14, 39, 45]).

For the DOMP framework, instead of providing a travel cost matrix between customers, we consider a travel distance measure  $d : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+$  induced by a norm  $\|\cdot\|$ , i.e.,  $d(a, b) = \|a - b\|$ , for  $a, b \in \mathcal{A}$ .

Also, each potential facility  $a \in \mathcal{A}$ , is associated to a convex set  $\mathcal{N}(a) \subset \mathbb{R}^d$ , with  $a \in \mathcal{N}(a)$ , its *neighbourhood*. We denote by  $\overline{\mathcal{N}} = \prod_{a \in \mathcal{A}} \mathcal{N}(a)$ , the space of neighborhoods. As in the DOMP case, set-up costs are provided for opening facilities (which may be neighborhood-dependent).

The goals of Discrete Ordered Median with Neighborhoods are:

- find the indices of the  $p$  facilities to open ( $\mathcal{B} = \{b_1, \dots, b_p\}$ ),
- locate the facilities into their neighbourhoods:  $\bar{b}_1, \dots, \bar{b}_p$  with  $\bar{b}_j \in \mathcal{N}(b_j)$ ,  $j = 1, \dots, p$ ,
- allocate customers to their closest open facilities among  $\bar{b}_1, \dots, \bar{b}_p$

by minimizing a discrete ordered median function of the travel distances and set-up costs.

Observe that the optimization problem to solve for DOMP is similar to (DOMP):

$$(\text{DOMPN}) \quad \min_{\substack{\mathcal{B} \subset \mathcal{A}: |\mathcal{B}|=p \\ \bar{a} \in \overline{\mathcal{N}}}} C(\mathcal{B}) := \sum_{i=1}^n \lambda_i D_{(i)} + \sum_{b \in \mathcal{B}} f(b)$$

but now,  $D_i = \min_{b \in \mathcal{B}} d(a_i, \bar{b})$ , i.e. the travel distance from a customer to its closest facility not only depends of the indices of the open facilities but also of the position of them inside their neighborhoods. So both the discrete location decisions and the continuous location of the new facilities decisions are involved in the problem (now the distances are not known).

Let us use the classical notation for the variables in  $p$ -median problems:

$$x_{ij} = \begin{cases} 1 & \text{if client } i \text{ is allocated to facility } j \text{ } (i \neq j) \text{ or if facility } j \text{ is open } (i = j), \\ 0 & \text{otherwise} \end{cases}$$

for  $i, j = 1, \dots, n$ .

Note that, using the above family of variables, the set of open facilities and assignments between customers and  $p$  facilities can be represented by the following set:

$$\begin{aligned} \mathcal{X} = \{x \in \{0, 1\}^{n \times n} : & \sum_{j=1}^n x_{ij} = 1, \forall i = 1, \dots, n, \\ & \sum_{j=1}^n x_{jj} = p, x_{ij} \leq x_{jj}, \forall i, j = 1, \dots, n\}, \end{aligned}$$

whose continuous relaxation  $\mathcal{X}_R = \{x \in [0, 1]^{n \times n} : \sum_{j=1}^n x_{ij} = 1, \forall i = 1, \dots, n, \sum_{j=1}^n x_{jj} = p, x_{ij} \leq x_{jj}, \forall i, j = 1, \dots, n\}$  is the so-called  *$p$ -median polytope*.

The set of distances, will be represented by the following set:

$$\mathcal{D} = \{(d, \bar{a}) \in \mathbb{R}_+^{n \times n} \times \bar{\mathcal{N}} : d_{ij} \geq \|a_i - \bar{a}_j\|, i, j = 1, \dots, n, i \neq j\},$$

where  $d_{ij}$  models the distance between customer located at  $a_i$  and the new facility located at  $\bar{a}_j$ , for all  $i, j = 1, \dots, n$ . Observe that the distance  $d_{ii}$  between a facility and its self is not considered into the distance set  $\mathcal{D}$ . This self-supply cost will be added into the objective function as part of the set-up costs. Note that the goal of DOMP is to locate the facilities to provided a minimum *ordered weighed sum* of travel costs of the customers, so sorting the set-up costs of the facilities as part of the travel costs has no sense under this framewok.

With the above notation, DOMP can be compactly formulated as:

$$(2.1) \quad \min \sum_{i=1}^n \lambda_i z_{(i)} + \sum_{j=1}^n f_j x_{jj}$$

$$(2.2) \quad \begin{aligned} \text{s.t. } z_i &= \sum_{j=1}^n d_{ij} x_{ij}, i, j = 1, \dots, n, \\ x &\in \mathcal{X}, (d, \bar{a}) \in \mathcal{D}. \end{aligned}$$

where  $f_j$  denotes the set-up cost of the facility initially located at  $a_j$ ,  $j = 1, \dots, n$  and  $z_i$  models the minimum distance between the  $i$ th customer and the opened facilities.

Observe that (2.1) is a mixed integer non linear programming problem (MINLP), whose continuous relaxation is not convex or concave due to the bilinear constraint (2.2) and probably to the constraints in  $\mathcal{D}$ . In case the neighborhoods are convex, the set  $\mathcal{D}$  is also convex. Hence, if the discrete location variables  $x$  were known, the problem (also because the convexity of the ordered median function), the problem would be a continuous convex problem (which in many cases, can be efficiently solved). On the other hand, if the distances were known, the problem, the problem become a DOMP with given distances, so several formulation can be applied to solve the problem. In the DOMP case, both  $\mathcal{X}$  and  $\mathcal{D}$  are part of the final decision. Thus, both the difficulties of handling the DOMP problem and the continuous problem are inherited to DOMP. In particular, since the  $p$ -median problem (or the  $p$ -center problem) is known to be NP-hard [23] which is a particular case of DOMP, the following result states the complexity of the problem under consideration.

**Theorem 2.1.** *DOMPN is NP-hard.*

The simplest DOMPN problem, apart from the DOMP case, is obtained when the set  $\mathcal{D}$  is defined by a set of linear inequalities. The difficulty of the set  $\mathcal{D}$  stems on two characteristics of the input information for DOMPN: the distance measure induced by  $\|\cdot\|$  and the shapes of the neighborhoods. The norms which are polyhedral-representable are called *block* (or polyhedral) norms (see [32]) are characterized by the fact that its unit ball is a polyhedron, i.e.,  $P = \{z \in \mathbb{R}^d : \|z\| \leq 1\}$  is a polyhedron. On the other hand, the neighborhoods, because they are assumed to be compact and convex sets, their polyhedral representability is assured if and only if they are polytopes. In those cases, both the set  $\mathcal{D}$  and  $\mathcal{X}$  are identified with sets of linear inequalities (and integrality constraints in  $\mathcal{X}$ ). Furthermore, as can be checked in [34] or [42], the ordered median function can be also modeled as a set of linear inequalities and equations by adding a set of  $n^2$  binary variables to our model. The above observations are summarized in the following result.

**Theorem 2.2.** *Let  $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ . If  $\|\cdot\|$  is a block norm and  $\mathcal{N}(a)$  is a polyhedron for each  $a \in \mathcal{A}$ , then DOMPN can be formulated a mixed-integer linear programming problem.*

*Proof.* The proof follows noting that constraints in the form  $Z \geq \|X - Y\|$ , as those that appear in the description of  $\mathcal{D}$ , can reformulated as:

$$Z \geq e^t(X - Y), \forall e \in \text{Ext}(P^*),$$

where  $\text{Ext}(P^*)$  the set of extreme points of  $P^* = \{v \in \mathbb{R}^d : v^t b_g \leq 1, g = 1, \dots, |\text{Ext}(P)|\}$ , the unit ball of the dual norm of  $P$  (see [32, 46]).  $\square$

The following example illustrates the new framework under consideration.

*Example 2.3.* Let us consider a set of customers/potential facilities with coordinates in the plane  $\mathcal{A} = \{(0, 5), (1, 1), (1, 6), (1, 4), (5, 3), (10, 4), (6.5, 0), (8, 6)\}$ , and travel distances measured by the Euclidean norm. The solutions of 2-median, 2-center and 2-4-center ( $K$ -center with  $K = 4$ ) are drawn in Figure 1 (stars represent open facilities and customers are identified with dots).

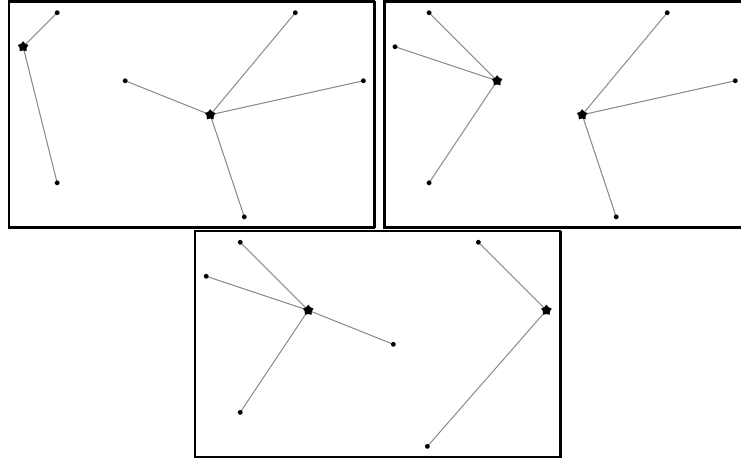


FIGURE 1. Solutions for 2-median, 2-center and 2-4-center for the data in Example 2.3.

Note that the solutions of DOMP problems depend of the  $\lambda$ -weights used. Being the optimal set of open facilities different for the three choices considered in the example.

Let us now consider, for each demand point, a neighbourhood defined as the disk with radii  $r \in \{(1, 0.6, 1, 0.6, 2.4, 2.4, 0.8, 1.6)\}$ . The neighborhoods are shown in Figure 2.

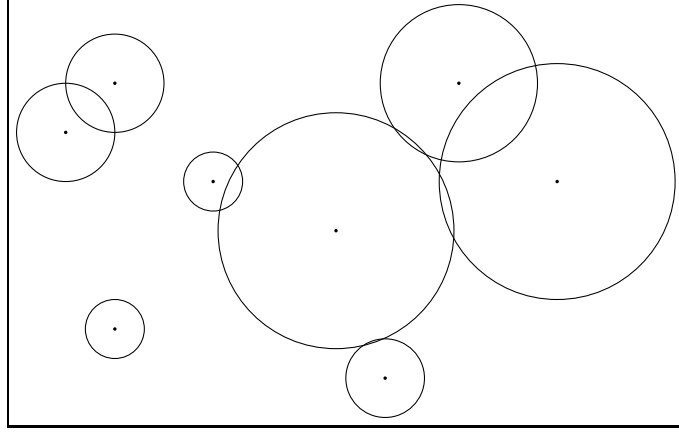


FIGURE 2. Neighbourhoods for the facilities of Example 2.3.

The new facilities are not restricted to be exactly located in the given coordinates but in a disk around them. We consider the set-up cost of each facility, the radius of its neighborhood (disk). The solutions of the neighbourhood version of the 2-median, 2-center and 2-4-center problems are shown in Figure 3 (again the stars symbols indicate optimal location in the neighborhoods and the dots the customers positions).

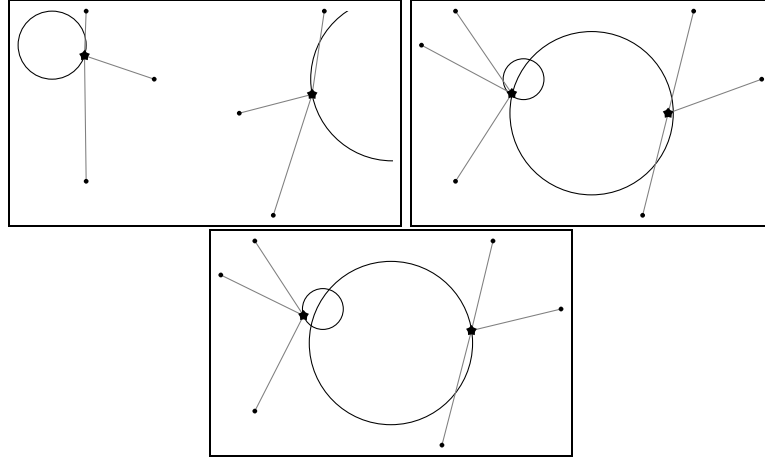


FIGURE 3. Solutions for 2-median, 2-center and 2-4-center with neighbourhood for the data in Example 2.3.

Note that when neighbourhoods around facilities are considered, both the open facilities and the allocation of customers may change with respect to the DOMP solutions.

Let us denote:

$$\widehat{D}_{ij} = \max\{\|a_i - \bar{a}_j\| : \bar{a}_j \in \mathcal{N}_j\}, \quad \text{and} \quad \widehat{d}_{ij} = \max\{\|a_i - \bar{a}_j\| : \bar{a}_j \in \mathcal{N}_j\}.$$

for  $i, j = 1, \dots, n$ , upper and lower bounds for the distances between customers and potential facilities.

Before describing some of the valid formulations for DOMP, some structural properties of DOMP that are inherited to DOMP are stated:

**Properties 2.4.** *The following properties are satisfied:*

- (1) *There exists an optimal solution of (DOMP) in which the  $p$  smaller travel distances equal 0.*
- (2) *Let  $\mathcal{B} \subseteq \mathcal{A}$  a set of  $p$  facilities such that its ordered cost  $C(\mathcal{B}) \leq UB$  and such that  $\min_{j \neq i} \hat{d}_{ij} > \frac{UB}{\sum_{i=1}^m \lambda_i}$  for some  $m = 2, \dots, n$ , then the  $i$ -th client is sorted at most in position  $m$  in the whole sorted set of optimal distances.*

*Proof.* (1) The property easily follows from the observation that if the facility  $a \in \mathcal{A}$  is selected as an open facility, the travel costs are assumed to be 0, and then sorted at the end of the non-increasing sequence of distances.

(2) Assume that the  $i$ th customer is sorted in position  $r \geq m$  in the sorting sequence of distances, i.e.,  $D_{(1)} \geq \dots \geq D_{(m)} \geq D_{(r)} = D_i$ . Then, we have that:

$$c(\mathcal{B}) = \sum_{l=1}^n \lambda_l D_{(l)} + \sum_{b \in \mathcal{B}} f(b) \geq \sum_{l=1}^m \lambda_l D_{(l)} \geq \sum_{l=1}^m \lambda_l D_i = D_i \left( \sum_{l=1}^m \lambda_l \right) > UB$$

which contradicts the hypothesis.

□

The above properties may help, by fixing to zero some of the binary variables that appear in some formulations of DOMP, to solve the problem.

### 3. MINLP FORMULATIONS FOR DOMP

In this section we describe different mathematical programming formulations for solving general DOMP. In particular, we formulate the problem by adapting the models presented in [5], [7] and [35]. As mentioned before the main difference between the DOMP and the DOMP problem is that in DOMP the distances are not part of the input information, but part of the decision itself. Hence, those formulations for the DOMP that are based on a preprocessing of the travel distances matrix (as those proposed in [25], [29], [28] or [37]) are not applicable to our framework, since the constraints indexation depend of the sorting sequence of distances which is not known in the DOMP case.

In the DOMP, the set  $\mathcal{D}$  conforms the *continuous location* part of the problem. An adequate representation of that set will be crucial for the development of efficient solution approaches for the problem. We will assume that the neighborhoods belongs to a family of convex sets that allows to represent most of the convex shapes which are useful in practice and that can be efficiently handled by most of the commercial optimization solvers: second order cone (SOC)-representable sets [27]. SOC-representable sets are convex sets defined by second-order cone constraints in the general form:

$$\|A_i x - b_i\|_2 \leq c_i^t x + d_i, \forall i = 1, \dots, M, x \in \mathbb{R}^N,$$

where  $A_1, \dots, A_i \in \mathbb{R}^{M_i \times N}$ ,  $b_i \in \mathbb{R}^{M_i}$ ,  $c_i \in \mathbb{R}^N$ ,  $d_i \in \mathbb{R}$ , for  $i = 1, \dots, M$ . Most of the state-of-the-art solvers are capable to efficiently solve optimization problems involving SOC constraints by means of quadratic constraints with positive definite matrices, second order cone (in the form  $x^t x \leq y^2$ , for  $y \geq 0$ ) or rotated second



order cones ( $x^t x \leq yz$  with  $y, z \geq 0$ ). SOC constraints allow to represent, not only Euclidean balls, but any  $\ell_\tau$ -norm ball (see [5] for further details on the explicit representation of  $\ell_\tau$ -norm based distance constraints as a set of SOC constraints for any  $\tau \in \mathbb{Q}$  with  $\tau \geq 1$ ). Clearly, any polyhedron is SOC-representable (setting  $A$  and  $b$  equal to zero) so any intersection of  $\ell_\tau$ -norms balls and polyhedra is suitable to be represented as a set of second order cone constraints. Hence, both our neighborhoods and the distances involved in our problem will be defined as SOC-constraints, being then  $\mathcal{D}$  a SOC-representable set.

For the sake of simplicity, and without loss of generality, we assume that the neighborhood of each  $a \in \mathcal{A}$  is a  $\ell_\tau$ -norm ball, i.e.  $\mathcal{N}(a) = \{z \in \mathbb{R}^d : \|z - a\|_\tau \leq r_a\}$ , for some  $r_a \in \mathbb{R}_+$  and  $\tau \in \mathbb{Q}$  with  $\tau \geq 1$ .

Also, we consider that the travel distances are induced by a  $\ell_\nu$ -norm with  $\nu \in \mathbb{Q}$  and  $\nu \geq 1$ . With these settings, we can explicitly describe  $\mathcal{D}$  as follows:

$$\mathcal{D} = \{(d, \bar{a}) \in \mathbb{R}_+^{n \times n} \times \mathbb{R}^{n \times d} : d_{ij} \geq \|a_i - \bar{a}_j\|_\nu, r_j \geq \|a_j - \bar{a}_j\|_\tau, i, j = 1, \dots, n\}$$

where, abusing of notation,  $r_j$  denotes the radius of the neighborhood  $\mathcal{N}(a_j)$ , i.e.,  $r_j := r_{a_j}$ .

The following result allows us to efficiently represent the set  $\mathcal{D}$  when the involved norms are  $\ell_\tau$ -based norms.

**Properties 3.1.** *Let  $\tau = \frac{r_\tau}{s_\tau} \geq 1$  and  $\nu = \frac{r_\nu}{s_\nu} \geq 1$  with  $r_\tau, s_\tau, r_\nu, s_\nu \in \mathbb{Z}_+$  and  $\gcd(r_\tau, s_\tau) = \gcd(r_\nu, s_\nu) = 1$ . Then,  $\mathcal{D}$  is representable as a set of  $(n^2 + n)(2d + 1)$  linear inequalities and  $nd(n \log r_\nu + \log r_\tau)$  quadratic constraints as second order cone constraints or rotated second order cone constraints.*

*Proof.* The proof follows by applying [5, Theorem 2], where the explicit representation is detailed.  $\square$

**3.1. The three index formulation.** The first formulation is based on the one proposed in [7], which uses, apart from the  $x_{jj}$ -variables described above, the following set of sorting/allocation binary variables for the DOMP:

$$w_{ij}^k = \begin{cases} 1 & \text{if customer } i \text{ is allocated to facility } j \text{ and its distance is sorted} \\ & \text{in the } k\text{th position.} \\ 0 & \text{otherwise.} \end{cases}$$

This formulation reads as follows:

$$\begin{aligned}
(\text{DOMPN}_{3I}) \quad & \min \sum_{i,j,k=1}^n \lambda_k d_{ij} w_{ij}^k + \sum_{j=1}^n f_j x_{jj} \\
(3.1) \quad & \text{s.t.} \quad \sum_{j,k=1}^n w_{ij}^k = 1, \forall i = 1, \dots, n, \\
(3.2) \quad & \sum_{i,j=1}^n w_{ij}^k = 1, \forall k = 1, \dots, n, \\
(3.3) \quad & \sum_{k=1}^n w_{ij}^k \leq x_{jj}, \forall i, j = 1, \dots, n, \\
(3.4) \quad & \sum_{j=1}^n x_{jj} = p, \\
(3.5) \quad & \sum_{i,j=1}^n d_{ij} w_{ij}^{k-1} \geq \sum_{i,j=1}^n d_{ij} w_{ij}^k, \forall k = 2, \dots, n, \\
& w_{ij}^k \in \{0, 1\}, \forall i, j, k = 1, \dots, n, \\
& x_{jj} \in \{0, 1\}, \forall j = 1, \dots, n. \\
& (d, \bar{a}) \in \mathcal{D}.
\end{aligned}$$

The objective function assign to each sorted distance its adequate weight  $\lambda$ . Observe that (3.1) (resp. (3.2)) assures that each client (resp. each position) is identified with an unique facility and position (resp. facility and client). (3.4) and (3.3) are the same as in classical  $p$ -median problems in which  $p$  new facilities have to be located and no allocation is allowed unless the plant is open. Constraints (3.5) allows a correct definition of the  $w$ -variables in which the sorting of the distances is imposed (the  $(k-1)$ th distance must be greater or equal than the  $k$ -th distance).

Although the above formulation is valid for DOMPN (note that there is no difference with respect to the one for DOMP except that the distances instead of being given are defined in  $\mathcal{D}$ ), both the objective function and the family of constraints (3.5) are quadratic and non-convex. We introduce a new set of variables to account for the non linear terms in the above formulation:

$$\theta_{ij}^k = d_{ij} w_{ij}^k, \quad i, j, k = 1, \dots, n.$$

Using the  $\theta$ -variables, the objective function can be reformulated as:

$$\sum_{i,j,k=1}^n \lambda_k \theta_{ij}^k + \sum_{j=1}^n f_j x_{jj},$$

and the correct definition of the new variables and satisfaction of constraints (3.5) is assured by the following linear constraints:

$$\begin{aligned}
\theta_{ij}^k &\geq d_{ij} - \hat{D}_{ij}(1 - w_{ij}^k), & \forall i, j, k = 1, \dots, n, \\
\sum_{i,j=1}^n \theta_{ij}^{k-1} &\geq \sum_{i,j=1}^n \theta_{ij}^k, & \forall i, j, k = 1, \dots, n.
\end{aligned}$$

where the first set of constraints comes from the McCormick linear reformulation [30] of the bilinear terms defining the  $\theta$ -variables, and the second is the reformulation of (3.5) with the new variables.

The formulation above, has  $O(n^3)$  variables and  $O(n^2)$  constraints. Properties 2.4 allow to strengthen the formulation ( $\text{DOMPN}_{3I}$ ). If  $UB$  is a known upper

bound for the optimal value of DOMP<sub>N</sub>, then

$$w_{ij}^k = 0, \forall i, j, k = 1, \dots, n, \text{ such that } \min_{j \neq i} \widehat{D}_{ij} > \frac{UB}{\sum_{l=k}^n \lambda_l}$$

and also:

$$\sum_{k=1}^n w_{jj}^k = x_{jj}, \forall j = 1, \dots, n$$

are valid equations for (DOMP<sub>N<sub>3I</sub></sub>).

**3.2. The 2-index formulation.** The second formulation, also based on the one presented in [7], considers an alternative representation of the sorting variables. It uses a different sets of variables. The first one allows to sort the distances of supplying each of the customers:

$$s_{ik} = \begin{cases} 1 & \text{if the distance supported by the } i\text{th customer is sorted in the } k \text{ position.} \\ 0 & \text{otherwise.} \end{cases}$$

while the second is the sorting (non decreasing) sequence of distances.

$$\xi_k = \sum_{i=1}^n s_{ik} \sum_{j=1}^n d_{ij} x_{ij}, \quad k = 1, \dots, n.$$

This representation allows to simplify the formulation to the following with  $O(n^2)$  variables and  $O(n^2)$  constraints.

$$(3.6) \quad \begin{aligned} \text{(DOMP}_{N_{2I}}) \quad & \min \sum_{k=1}^n \lambda_k \xi_k + \sum_{j=1}^n f_j x_{jj} \\ & \text{s.t. } \xi_k \geq \xi_{k+1}, \forall k = 1, \dots, n-1, \end{aligned}$$

$$(3.7) \quad \sum_{k=1}^n \xi_k = \sum_{i,j=1}^n d_{ij} x_{ij},$$

$$(3.8) \quad \xi_k \geq d_{ij} x_{ij} - \widehat{D}_{ij} (1 - s_{ik}), \forall i, k = 1, \dots, n,$$

$$(3.9) \quad \sum_{i=1}^n s_{ik} = 1, \forall k = 1, \dots, n$$

$$(3.10) \quad \begin{aligned} & \sum_{k=1}^n s_{ik} = 1, \forall i = 1, \dots, n, \\ & \xi_i \geq 0, \forall i, k = 1, \dots, n, \\ & s_{ik} \in \{0, 1\}, \forall i, k = 1, \dots, n, \\ & x \in \mathcal{X}, \\ & (d, \bar{a}) \in \mathcal{D}. \end{aligned}$$

The correct definition of the  $\xi$ -variables is assured by constraints (3.6)–(3.8), while constraints (3.9) and (3.10) allows the adequate modeling of the  $s$ -variables.

As in (DOMP<sub>N<sub>3I</sub></sub>), to avoid the appearance of nonconvex terms in the formulation, the bilinear terms  $d_{ij} x_{ij}$  can be linearized by introducing a new variable

$\theta_{ij} = d_{ij}x_{ij}$  and replacing (3.7) and (3.8) by:

$$(3.11) \quad \sum_{k=1}^n \xi_k = \sum_{i,j=1}^n \theta_{ij},$$

$$(3.12) \quad \xi_k \geq \theta_{ij} - \widehat{D}_{ij} (1 - s_{ik}), \forall i, j, k = 1, \dots, n.$$

$$(3.13) \quad \theta_{ij} \geq d_{ij} - \widehat{D}_{ij} (1 - x_{ij}), \forall i, j, k = 1, \dots, n.$$

**3.3. The  $K$ -sum formulation.** Ogryczak and Tamir presented in [35] some linear programming formulations for the problem of minimizing the sum of the  $K$  largest linear functions (which is a particular case of ordered median function). In the same paper, the approach is extended to the minimization of convex ordered median functions by means of a telescopic sum of  $K$ -sum functions. In the next formulation, we apply this idea to formulate DOMP. For the sake of readability, we first formulate the  $K$ -center problem and we will provide a valid formulation form DOMP as a consequence of it.

Let us consider  $\lambda = (\overbrace{1, \dots, 1}^K, \overbrace{0, \dots, 0}^{n-K})$ . The ordered median function associated to this particular choice of  $\lambda$  is known as the  $K$ -center function, and minimizing it consists of minimizing the sum of the  $K$  largest values. With our notation, provided the set of distances  $D_1, \dots, D_n$ , the  $K$ -center problem consists of minimizing  $\Theta_K(D) = \sum_{i=1}^K D_{(i)}$ . Such an objective function, is proved in [35] to be equivalent to the following expression

$$\Theta_K(D) = \frac{1}{n} \left( K \sum_{i=1}^n D_i + \min_{t \in \mathbb{R}} \sum_{i=1}^n (K (t - D_i)_+ + (n - K) (D_i - t)_+) \right)$$

where  $z_+ = \max\{0, z\}$  for  $z \in \mathbb{R}$ , and the optimal value  $t^*$  into the above expression coincides with  $D_{(K)}$ . Hence, in order to minimize the above expression, one may proceed by solving:

$$\begin{aligned} \min \quad & K t + \sum_{i=1}^n z_i \\ \text{s.t.} \quad & z_i \geq D_i - t, \forall i = 1, \dots, n, \\ & z_i \geq 0, \forall i = 1, \dots, n, \\ & t \in \mathbb{R}. \end{aligned}$$

where the variable  $z_i$  represents  $(D_i - t)_+$  in the above formulation, for  $i = 1, \dots, n$ . Thus, incorporating the whole information to define the distances in our location problem, the  $K$ -center discrete location problem with neighborhoods can be formulated as:

$$(KCN_{OT}) \quad \min \quad K t + \sum_{i=1}^n z_i + \sum_{j=1}^n f_j x_{jj}$$

$$(3.14) \quad \text{s.t.} \quad z_i \geq D_i - t, \forall i = 1, \dots, n,$$

$$(3.15) \quad D_i \geq d_{ij} - \widehat{D}_{ij}(1 - x_{ij}), \forall i, j = 1, \dots, n,$$

$$(3.16) \quad z_i, D_i \geq 0, \forall i = 1, \dots, n,$$

$$(3.17) \quad t \in \mathbb{R},$$

$$x \in \mathcal{X},$$

$$(d, \bar{a}) \in \mathcal{D}.$$

where  $D_i = \sum_{j=1} d_{ij}x_{ij}$  allows to determine the closest distance from a customer  $i$  to an open facility (constraint (3.15)).

The above formulation can be extended to general convex ordered median functions. Observe that if  $\lambda_1 \geq \dots \geq \lambda_n \geq \lambda_{n+1} := 0$  one may represent the ordered median function of our given distances  $D_1, \dots, D_n$  as:

$$\begin{aligned} \sum_{i=1}^n \lambda_i D_{(i)} &= [(\lambda_1 - \lambda_2) + (\lambda_2 - \lambda_3) + \dots + (\lambda_n - \lambda_{n+1})] d_{(1)} \\ &+ [(\lambda_2 - \lambda_3) + \dots + (\lambda_n - \lambda_{n+1})] D_{(2)} + \\ &\vdots \\ &+ (\lambda_n - \lambda_{n+1}) D_{(n)} \\ &= \sum_{K=1}^n (\lambda_K - \lambda_{K+1}) \sum_{i=1}^K D_{(i)} \end{aligned}$$

Then, the convex ordered objective function can be represented as a weighted sum of  $K$ -sums, so suitable to be represented as in the  $K$ -center problem. With such an observation and introducing new  $t$ -variables (one for each of the  $K$ -sums involved) and  $z$ -variables, in this case with two indices to account not only for the customer ( $i$ ) but also for the  $K$ -sum representation, one obtain the following valid formulation for the DOMP<sub>N</sub>:

$$\begin{aligned} (\text{DOMP}_{N_{OT}}) \quad & \min \sum_{k=1}^n (\lambda_k - \lambda_{k+1}) (kt_k + \sum_{i=1}^n z_{ik}) + \sum_{j=1}^n f_j x_{jj} \\ \text{s.t.} \quad & z_{ik} \geq D_i - t_k, \forall i, k = 1, \dots, n, \\ (3.18) \quad & D_i \geq d_{ij} - \widehat{D}_{ij}(1 - x_{ij}), \forall i, j = 1, \dots, n, \\ (3.19) \quad & z_{ik}, D_i \geq 0, \forall i = 1, \dots, n, \\ (3.20) \quad & t_k \in \mathbb{R}, k = 1, \dots, n, \\ & x \in \mathcal{X}, \\ & (d, \bar{a}) \in \mathcal{D}. \end{aligned}$$

Observe that this formulation has also  $O(n^2)$  variables and  $O(n^2)$  constraints, but, as will be shown in the computational results, it has a better performance than (DOMP<sub>N<sub>2I</sub></sub>) since it uses intrinsically the especial structure of the convex ordered median objective. A family of valid inequalities for the above formulations is

$$D_i \leq \widehat{D}_{ij}x_{ij}, \forall i, j = 1, \dots, n,$$

which avoid searching for the distances  $D_i$  between pairs of customers and facilities which are not allocated.

**3.4. The BEP formulation.** Finally, we present a formulation, based on the one provided in [5] and which, as (DOMP<sub>N<sub>OT</sub></sub>), is valid only for the convex case. The idea under the formulation comes from the observation that, since  $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ , the value of a ordered median is always reached when choosing, among all the possible permutation of the indices,  $\mathcal{P}_n$ , the one that maximizes the weighted sum, i.e.,  $\sum_{i=1}^n \lambda_i D_{(i)} = \max_{\sigma \in \mathcal{P}_n} \sum_{i=1}^n \lambda_i D_{\sigma(i)}$ . The set of permutations of  $\{1, \dots, n\}$  can be represented as a set of binary variables  $p_{ik} =$

$\begin{cases} 1 & \text{if the permutation assigns index } i \text{ to index } k, \\ 0 & \text{otherwise} \end{cases}$ , verifying that  $\sum_{i=1}^n p_{ik} = 1$  (for all  $k = 1, \dots, n$ ) and  $\sum_{k=1}^n p_{ik} = 1$  (for all  $i = 1, \dots, n$ ). Thus, the ordered median sum of a given set of values  $D_1, \dots, D_n$  is equivalent to:

$$\begin{aligned} \sum_{i=1}^n \lambda_i D_{(i)} &= \max_{p \in \{0,1\}^{n \times n}} \sum_{i,k=1}^n \lambda_k D_i p_{ik} \\ \text{s.t. } &\sum_{i=1}^n p_{ik} = 1, \forall k = 1, \dots, n, \\ &\sum_{k=1}^n p_{ik} = 1, \forall i = 1, \dots, n. \end{aligned}$$

Since the optimization problem above is an assignment problem, the total unimodularity of the constraints matrix assures that its optimal value coincides with the one of its dual which read:

$$\begin{aligned} \min &\sum_{k=1}^n u_k + \sum_{i=1}^n v_i \\ \text{s.t. } &u_i + v_k \geq \lambda_k D_i, \forall i, k = 1, \dots, n, \end{aligned}$$

Thus, since now the problem is in minimization form (as our original problem), the (dual) ordered median function above can be embedded in the original DOMP<sub>N</sub>, which can be now equivalently formulated as:

$$\text{(DOMPN}_{BEP}) \quad \min \sum_{k=1}^n u_k + \sum_{i=1}^n v_i + \sum_{j=1}^n f_j x_{jj}$$

$$(3.21) \quad \text{s.t. } u_i + v_k \geq \lambda_k D_i, \forall i, k = 1, \dots, n,$$

$$\begin{aligned} (3.22) \quad &D_i \geq d_{ij} - \widehat{D}_{ij}(1 - x_{ij}), \forall i, j = 1, \dots, n, \\ &x \in \mathcal{X}, \\ &(d, \bar{a}) \in \mathcal{D}. \end{aligned}$$

where again (3.22) is the McCormick linearization of the bilinear term  $D_i = \sum_{j=1}^n d_{ij} x_{ij}$ . Valid inequalities as for the previous formulation are possible:

$$D_i \leq \widehat{D}_{ij} x_{ij}, \forall i, j = 1, \dots, n$$

**3.5. Comparison of Formulations.** In this section we state some theoretical results comparing the tightening of the convex relaxations of each of the provided formulations. Hence, we denote by  $F_{3I}$ ,  $F_{2I}$ ,  $F_{OT}$  and  $F_{BEP}$  the feasible regions of formulations (DOMPN<sub>3I</sub>), (DOMPN<sub>2I</sub>), (DOMPN<sub>OT</sub>) and (DOMPN<sub>BEP</sub>), respectively obtained by relaxing the integrality conditions of the location-allocation variables. We also denote by  $z_{3I}^R$ ,  $z_{2I}^R$ ,  $z_{OT}^R$  and  $z_{BEP}^R$  the optimal objective values of the continuous relaxation of each of the formulations.

**Theorem 3.2.** *The following relations are verified:*

- (1)  $z_{3I}^R \leq z_{2I}^R$ .
- (2)  $z_{2I}^R \leq z_{BEP}^R$ .
- (3)  $z_{OT}^R = z_{BEP}^R$ .

*Proof.* (1) Let us consider the mapping  $\pi : \mathbb{R}_+^n \times \mathbb{R}_+^{n \times n} \times [0, 1]^{n \times n} \times \mathcal{X}_R \times \mathcal{D} \rightarrow \mathbb{R}_+^{n^3} \times [0, 1]^3 \times [0, 1]^n \times \mathcal{D}$  defined as:

$$\pi(\xi, \theta, s, x, (d, \bar{a})) = ((\xi_k s_{ik} x_{ij})_{i,j,k=1}^n, (s_{ik} x_{ij})_{i,j,k=1}^n, (x_{jj})_{j=1}^n, (d, \bar{a}))$$

which *projects* the feasible set of  $F_{2I}$  into  $F_{3I}$ .

Let us check that  $\pi(F_{2I}) \subseteq F_{3I}$  proving the result. Let us consider a feasible element in  $(\theta, \xi, s, x, (d, \bar{a})) \in F_{2I}$ , and define  $(\bar{\theta}, \bar{x}, (d, \bar{a})) = \pi(\theta, \xi, s, x, (d, \bar{a}))$ , i.e.:

$$\bar{\theta}_{ij}^k = \xi_k s_{ik} x_{ij}, \quad \bar{w}_{ij}^k = s_{ik} x_{ij}, \quad \bar{x}_{jj} = x_{jj}, \quad \forall i, j, k = 1, \dots, n.$$

By construction, the constraints (3.1)-(3.4) are verified:

$$\begin{aligned} \bullet & \sum_{j,k=1}^n \bar{w}_{ij}^k = \sum_{j,k=1}^n s_{ik} x_{ij} = \sum_{j=1}^n x_{ij} \sum_{k=1}^n s_{ik} \stackrel{(3.10)}{=} \sum_{j=1}^n x_{ij} \stackrel{x \in \mathcal{X}_R}{=} 1. \\ \bullet & \sum_{i,j=1}^n \bar{w}_{ij}^k = \sum_{i,j=1}^n s_{ik} x_{ij} = \sum_{i=1}^n s_{ik} \sum_{j=1}^n x_{ij} \stackrel{x \in \mathcal{X}_R}{=} \sum_{i=1}^n s_{ik} \stackrel{(3.9)}{=} 1. \\ \bullet & \sum_{k=1}^n \bar{w}_{ij}^k = \sum_{k=1}^n s_{ik} x_{ij} = x_{ij} \sum_{k=1}^n s_{ik} \stackrel{(3.10)}{=} x_{ij} \stackrel{x \in \mathcal{X}_R}{=} x_{jj}. \\ \bullet & \sum_{j=1}^n \bar{x}_{jj} = \sum_{j=1}^n x_{jj} \stackrel{x \in \mathcal{X}_R}{=} p. \end{aligned}$$

Let us now check the constraints derived of the linearization of constraint (3.5):

$$\begin{aligned} \bullet & \bar{\theta}_{ij}^k = \xi_k s_{ik} x_{ij} \stackrel{(3.12), (3.13)}{\geq} (d_{ij} - \hat{D}_{ij}(2 - s_{ik} - x_{ij})) s_{ik} x_{ij} = d_{ij} - \hat{D}_{ij}(1 - s_{ik} x_{ij}) + (\hat{D}_{ij} - d_{ij})(1 - s_{ik} x_{ij}) + \hat{D}_{ij} s_{ik} x_{ij} (s_{ik} + x_{ij} \stackrel{\bar{w}_{ij}^k = s_{ik} x_{ij}, s_{ik}, x_{ij} \leq 1, d_{ij} \leq \hat{D}_{ij}}{\geq} \\ & d_{ij} - \hat{D}_{ij}(1 - \bar{w}_{ij}^k)). \\ \bullet & \sum_{i,j=1}^n \bar{\theta}_{ij}^k = \sum_{i,j=1}^n \xi_k s_{ik} x_{ij} = \xi_k \sum_{i=1}^n s_{ik} \sum_{j=1}^n x_{ij} \stackrel{x \in \mathcal{X}_R}{=} \xi_k \sum_{i=1}^n s_{ik} \stackrel{(3.9)}{=} \xi_k \stackrel{(3.6)}{\geq} \\ & \xi_{k+1} = \sum_{i,j=1}^n \bar{\theta}_{ij}^{k+1}. \end{aligned}$$

Thus,  $\pi(\theta, \xi, s, x, (d, \bar{a})) \in F_{3I}$ , for any  $(\theta, \xi, s, x, (d, \bar{a})) \in F_{2I}$ , so  $\pi(F_{2I}) \subseteq F_{3I}$ . Any solution of the convex relaxation of (DOMPN<sub>2I</sub>) induces a solution of the convex relaxation of (DOMPN<sub>3I</sub>). Furthermore, the objective values for  $(\theta, \xi, s, x, (d, \bar{a}))$  in (DOMPN<sub>2I</sub>) and  $\pi(\theta, \xi, s, x, (d, \bar{a}))$  in (DOMPN<sub>3I</sub>) coincides:

$$\begin{aligned} \sum_{i,j,k=1}^n \lambda_k \bar{\theta}_{ij}^k + \sum_{j=1}^n f_j \bar{x}_{jj} &= \sum_{i,j,k=1}^n \lambda_k \xi_k s_{ik} x_{ij} + \sum_{j=1}^n f_j x_{jj} \\ &= \sum_{k=1}^n \lambda_k \xi_k \sum_{i=1}^n s_{ik} \sum_{j=1}^n x_{ij} \\ &\stackrel{x \in \mathcal{X}_R}{=} \sum_{k=1}^n \lambda_k \xi_k \sum_{i=1}^n s_{ik} \\ &\stackrel{(3.9)}{=} \sum_{k=1}^n \lambda_k \xi_k \end{aligned}$$

Thus,  $z_{2I}^R \geq z_{3I}^R$ .

(2) Let  $(u, v, D, x, (d, \bar{a})) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+^n \times \mathcal{X}_R \times \mathcal{D}$  be the optimal solution of the continuous relaxation of (DOMPN<sub>BEP</sub>). Let  $p_{ik}$  the optimal dual

variables associated to constraint (3.21). By optimality conditions they must verify:

$$\begin{aligned} \sum_{i=1}^n p_{ik} &= 1, \forall k = 1, \dots, n, \\ \sum_{k=1}^n p_{ik} &= 1, \forall i = 1, \dots, n. \end{aligned}$$

Let us construct the following vector in  $\mathbb{R}_+^n \times \mathbb{R}_+^{n \times n} \times [0, 1]^{n \times n} \times \mathcal{X}_R \times \mathcal{D}$ :

$$(\bar{\xi}, \bar{\theta}, \bar{s}, x, (d, \bar{a})) := \left( \left( \sum_{i=1}^n p_{ik} D_i \right)_{k=1}^n, (d_{ij} x_{ij})_{i,j=1}^n, (p_{ik})_{i,k=1}^n, x, (d, \bar{x}) \right).$$

By construction,  $\bar{s}$  clearly verifies (3.9) and (3.10). Also, note from the construction of the BEP formulation that for the given  $D_1, \dots, D_n$ , the problem

$$\begin{aligned} \min \quad & \sum_{i=1}^n u_i + \sum_{k=1}^n v_k \\ \text{s.t.} \quad & u_i + v_k \geq D_i \quad u_i, v_k \in \mathbb{R}, \forall i, k = 1, \dots, n. \end{aligned}$$

is equivalent to

$$\begin{aligned} \max \quad & \sum_{i,k=1}^n \lambda_k D_i p_{ik} \\ \text{s.t.} \quad & \sum_{i=1}^n p_{ik} = 1, \forall k = 1, \dots, n, \\ & \sum_{k=1}^n p_{ik} = 1, \forall i = 1, \dots, n. \\ & p_{ik} \in \{0, 1\} \end{aligned}$$

which is an assignment problem related to the *best* sorting on the variables based on their costs given by  $D_1, \dots, D_n$ . Because the monotonicity and nonnegativity of the  $\lambda$ -weights, this is equivalent to compute the ordered

median sum  $\sum_{i=1}^n \lambda_i D_{(i)} = \sum_{i,k=1}^n \lambda_k D_i p_{ik}$  (where  $p$  are the corresponding so-

lution to the problem above indicating if  $p_{ik} = 1$  that element  $i$  is sorted in the  $k$ th position). Hence,  $\bar{\xi}_k = \sum_{i=1}^n p_{ik} D_i \geq \sum_{i=1}^n p_{i,k+1} D_i = \bar{\xi}_{k+1}$  (constraint

(3.6)). The proof of the verification of remainder constraints are straightforward. Also, the reader can easily check that the objective values of both solutions coincide. Thus,  $z_{BEP}^R \geq z_{3I}^R$

- (3) Let  $(u, v, D, x, (d, \bar{a})) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+^n \times \mathcal{X}_R \times \mathcal{D}$  be the optimal solution of the continuous relaxation of (DOMPN<sub>BEP</sub>). Let us construct a feasible solution for the continuous relaxation of (DOMPN<sub>OT</sub>). Let  $(\bar{t}, \bar{z}) \in \mathbb{R}^n \times$



$\mathbb{R}_+^{n \times n}$  the solution to the problem

$$\begin{aligned} \min \quad & \sum_{k=1}^n (\lambda_k - \lambda_{k+1}) (kt_k + \sum_{i=1}^n z_{ik}) \\ \text{s.t.} \quad & z_{ik} \geq D_i - t_k, \forall i, k = 1, \dots, n, \\ & z_{ik} \geq 0, \forall i, k = 1, \dots, n, \\ & t_k \in \mathbb{R}, \forall k = 1, \dots, n. \end{aligned}$$

By the construction in Subsection 3.3, the vector  $(\bar{t}, \bar{z}, D, x, (d, \bar{a}))$  is a feasible solution to the continuous relaxation of  $(\text{DOMPN}_{OT})$  with same objective value than  $(u, v, D, x, (d, \bar{a}))$  in the continuous relaxation of  $(\text{DOMPN}_{BEP})$ , being then  $z_{OT}^R \leq z_{BEP}^R$ . Observe that the opposite direction can be derived with a similar reasoning. Thus,  $z_{OT}^R = z_{BEP}^R$ .  $\square$

In the above theorem is not stated the relation between  $z_{2I}^R$  and  $z_{BEP}^R$  (or  $z_{OT}^R$ ). One may think that  $(\text{DOMPN}_{BEP})$  is stronger (by means of the continuous relaxation) than  $(\text{DOMPN}_{2I})$ . However, that is not always true as illustrated in the following example.

*Example 3.3.* Let us consider five points in  $\mathbb{R}^2$ ,  $\mathcal{A} = \{(2, 92), (33, 70), (65, 50), (73, 69), (40, 63)\}$  and neighborhoods defined as Euclidean disks with radii  $\{2, 1, 0.05, 5, 1\}$ . If the distance measure is also the Euclidean norm, we get that:

$$z_{3I}^R = 2.8348 < z_{OT}^R = z_{BEP}^R = 24.4140 < z_{2I}^R = 34.2145 < z^* = 69.4236$$

We have run a series of experiments to study the computational performance of the formulations  $(\text{DOMPN}_{3I})$ ,  $(\text{DOMPN}_{2I})$ ,  $(\text{DOMPN}_{OT})$  and  $(\text{DOMPN}_{BEP})$  and also to know the size limits of the DOMPN. We have randomly generated instances of  $n$  demand points in  $[0, 100]^2$  and  $[0, 100]^3$  with  $n$  ranging in  $\{5, \dots, 10\}$ . Five instances were generated for each number of points. DOMPN problems were solved to locate  $p$  facilities with  $p \in \{2, 3, 5\}$  ( $p < n$ ). Euclidean distances were considered to measure the distances between points. The neighborhoods were considered as Euclidean balls with randomly generated radius, by considering four different scenarios:

**Scenario 1.:** Radii generated in  $[0, 5]$ .

**Scenario 2.:** Radii generated in  $[5, 10]$ .

**Scenario 3.:** Radii generated in  $[10, 15]$ .

**Scenario 4.:** Radii generated in  $[15, 20]$ .

In Figure 4 we show, for the same set of points their neighborhoods for the four scenarios.

The set-up cost of each facility were fixed to the radius of its neighborhood, i.e.  $f_j = r_j$ , for  $j = 1, \dots, n$ , which can be seen as the cost of covering the neighborhood (larger as  $r$  increases). The four formulations were coded in C, and solved using Gurobi 7.01 in a Mac OSX El Capitan with an Intel Core i7 processor at 3.3 GHz and 16GB of RAM. A time limit of 1 hour was set in all the experiments.

Also, four different convex ordered median problems were solved for each of the instances:

**$p$ -Median (M):**  $\lambda = (1, \dots, 1)$ .

**$p$ -Center (C):**  $\lambda = (1, 0, \dots, 0)$ .

**$p$ -KCenter (K):**  $\lambda = (\overbrace{1, \dots, 1}^{\lfloor \frac{n}{2} \rfloor}, 0, \dots, 0)$ .

**$p$ -Cent-Dian<sub>0.5</sub> (D):**  $\lambda = (1, 0.5, \dots, 0.5)$ .

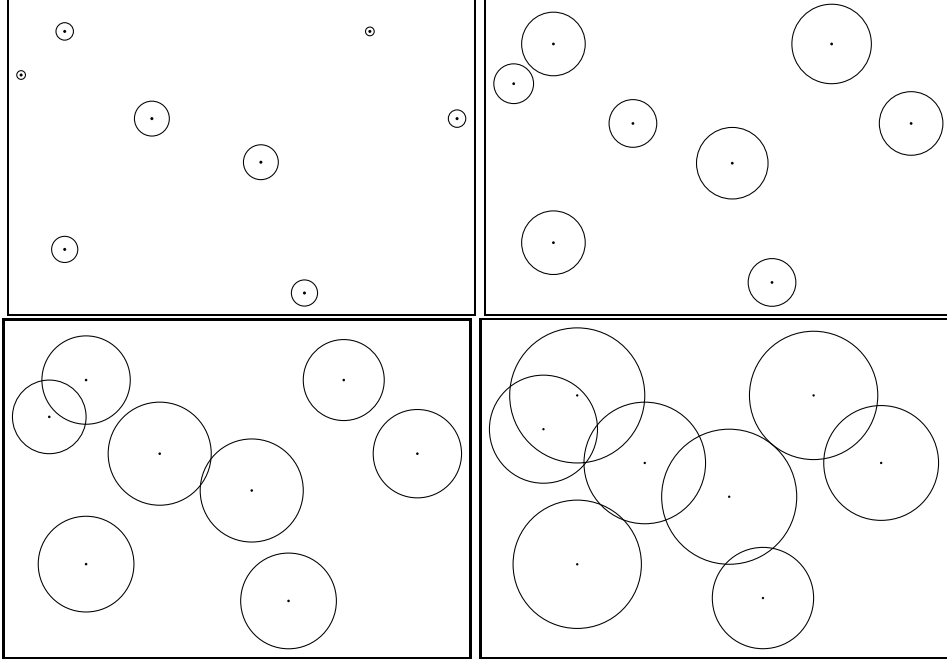


FIGURE 4. Shapes of the neighborhoods for the different scenarios.

First, we run the continuous relaxation of the four formulations to analyze the integrality gaps obtained. The average results are shown in Table 2. There, we show the integrality gaps ( $IG = \frac{z^*}{z^R}$ ) for each of the formulations ((DOMPN<sub>3I</sub>), (DOMPN<sub>2I</sub>) and (DOMPN<sub>BEP</sub>) (obviating (DOMPN<sub>OT</sub>) whose integrality gap coincides with the one of (DOMPN<sub>BEP</sub>) by Theorem 3.2). The table summarizes the average integrality gaps for the each of the scenarios. As remarked above, there is no order relation between  $z_{2I}^R$  and  $z_{BEP}^R$ . We have boldfaced those values in which the average integrality gaps of (DOMPN<sub>2I</sub>) is smaller than the one for (DOMPN<sub>BEP</sub>). Note that it only happens for a few combinations of  $n$ ,  $p$ , problem types and scenarios. In particular, it mostly occurs for small values of  $p$  and only for Scenario 1. In Table 3 we show the percentage of instances (of all of those with fixed  $n$  and  $p$ ) for which  $z_{2I}^R \geq z_{BEP}^R$ . In the total instances, this percentage is 9.97%, while among the instances of Scenario 1 is 26.15%.

Tables 4-7 show the results for the planar problems and Tables 8-11 show the results for the 3-dimensional problems. We only report the results obtained for  $n = 10$  (the interested reader may check the complete result of the experiments in [bit.ly/resultsexactDOMPN](http://bit.ly/resultsexactDOMPN)). For each of the formulations, we provide the average consumed times (**time**), the number of nodes explored in the branch-and-bound tree (**#nodes**) and the deviation with respect to the solution obtained at the end of the root node of the search tree (**%gapR**).

As can be observed from the results, formulations (DOMPN<sub>OT</sub>) and (DOMPN<sub>BEP</sub>) are much less time consuming than (DOMPN<sub>3I</sub>) and (DOMPN<sub>2I</sub>) in all the cases. Also, the continuous relaxations of (DOMPN<sub>OT</sub>) and (DOMPN<sub>BEP</sub>) are tighter than the other two. Consequently, the number of explored nodes to find the optimal solution or to certify optimality is higher in the two first formulations. The results were somehow expected since the formulations proposed in [7] were designed for general ordered median problems while (DOMPN<sub>OT</sub>) and (DOMPN<sub>BEP</sub>) exploit the convexity of the ordered median function when the

$\lambda$ -weights are monotone. Observe that the *sorting constraints* in the first two formulations involve binary variables while in the two last formulations no need of new binary variables are needed for this task.

Since (DOMPN<sub>OT</sub>) and (DOMPN<sub>BEP</sub>) seems to have a similar computational behaviour for the small-size instances, we have performed a series of experiments for larger instances to compare these two formulations. The results are shown in Table 12, where now  $n \in \{20, 30\}$  and  $p \in \{2, 5, 10\}$ . As can be observed, the performance (in terms of CPU time) of both formulation is similar, but (DOMPN<sub>BEP</sub>) seems to need, in average, less time to solve the problems in most of the problems, and the standard deviations (StDev) of the consuming times for (DOMPN<sub>BEP</sub>) is smaller than the obtained for (DOMPN<sub>OT</sub>), so more stable times are obtained. Also, (DOMPN<sub>BEP</sub>) was able to solve all the instances before the 1 hour time limit, but 2.56% of them, while (DOMPN<sub>OT</sub>) did not solved 11.34% of the instances. Moreover, in all the instances, (DOMPN<sub>BEP</sub>) obtained better upper bounds for the optimal value of the problems in all the instances (the deviation of the best upper bounds obtained with the OT formulation with respect to the best solution obtained with the BEP formulation is shown in column %DevBest).

#### 4. MATH-HEURISTICS FOR DOMPN

In this section we describe two mathematical programming based heuristic approaches for solving DOMPN for larger instances. Some heuristics have been proposed for solving ordered  $p$ -median problems (see [14]). However, most of them are based on the use of “fast” procedures to compute the overall cost of opening/closing certain sets of facilities. Note that when the set of allocation costs is given and a set of open facilities is provided, one can easily evaluate, for each customer, its cheapest facility (or its second cheapest facility), and once all of them are computed, evaluate an ordered median function can be also efficiently performed. In the DOMPN case, even if the open facilities are known, the allocation costs depend of the final location of the facilities (which depend of the customers allocated to each facility). Hence, the known heuristics are no longer valid for the DOMPN problem. We propose two alternative local search math-heuristic which allows to solve larger instances of the problem at smaller computational costs than the exact approaches, at the price of not warranting the optimality of the solution.

The two heuristics procedures that we propose are based on perform changes over a set of  $p$  facilities candidates for being opened, trying to improve the incumbent objective value. For the sake of that we need to compute the overall cost of opening a given set of facilities  $J \subseteq \{1, \dots, n\}$  with  $|J| = p$ , i.e., if the set of open facilities is known, the minimum (ordered weighted) cost of allocating the customers to those facilities. As mentioned above the computation of such a cost will involve the computation of the allocation customer-open facility and also the position of the open facilities inside their neighborhoods. Although different formulations can be used for such a task, we will use the one based on formulation (DOMPN<sub>BEP</sub>) which was tested in the previous section to provide one of the best performances.

**Theorem 4.1.** *Let  $J \subset N := \{1, \dots, n\}$  with  $|J| = p$ . Then, the cost of using  $J$  as open facilities of the DOMPN problem can be computed by solving the following mixed integer non linear programming problem:*

$$\begin{aligned}
(\text{ALLOC}(J)) \quad c(J) := & \min \sum_{i \in N \setminus J} u_i + \sum_{k \in N \setminus J} v_k + \sum_{j \in J} f_j, \\
\text{s.t. } & u_i + v_k \geq \lambda_k z_i, \forall i, k \in N \setminus J, \\
& z_i \geq d_{ij} - \widehat{D}_{ij}(1 - x_{ij}), \forall i \in N \setminus J, j \in J, \\
& d_{ij} \geq \|a_i - \bar{a}_j\|, \forall i \in N \setminus J, j \in J, \\
& r_j \geq \|a_j - \bar{a}_j\|, \forall j \in J, \\
& \sum_{j \in J} x_{ij} = 1, \forall i \in N \setminus J, \\
& x_{ij} \in \{0, 1\}, \forall i \in N \setminus J, j \in J, \\
& z_i \geq 0, \forall i \in N \setminus J.
\end{aligned}
\tag{4.1}$$

*Proof.* The proof easily follows noting that  $(\text{ALLOC}(J))$  is nothing but the simplification of  $(\text{DOMPN}_{BEP})$  when the values of  $x_{jj}$  are known and fixed to  $x_{jj} = 1$  if  $j \in J$  and  $x_{jj} = 0$ , otherwise.  $\square$

For each  $J$ ,  $(\text{ALLOC}(J))$  can be reformulated as a mixed integer second order cone constraint problem with  $(n - p)p$  binary variables (instead of the  $n^2$  in  $(\text{DOMPN}_{BEP})$ ). Furthermore, a variable fixing strategy can be applied to  $(\text{ALLOC}(J))$  in order to reduce the number of binary variables of the problem.

**Properties 4.2.** *Let  $J \subseteq N$  with  $|J| = p$ ,  $i \in N \setminus J$ ,  $j \in J$  and  $x^* \in \mathcal{X}$  optimal allocation solutions of  $(\text{ALLOC}(J))$ .*

- (1) *If  $\exists k \in J \setminus \{j\}$  such that  $\widehat{D}_{ik} < \widehat{d}_{ij}$ , then  $x_{ij}^* = 0$ .*
- (2) *If  $\min_{k \neq j} \widehat{d}_{ik} > \widehat{D}_{ij}$ , then  $x_{ij}^* = 1$ .*
- (3) *If  $\{j' \in J : \widehat{D}_{ik} \geq \widehat{d}_{ij'}, \forall k \neq j'\} = \{j\}$ , then  $x_{ij}^* = 1$ .*

*Proof.*

- (1) Let us assume that  $x_{ij}^* = 1$ . Then,  $d_{ij}^* = \|a_i - \bar{a}_j^*\| \geq \widehat{d}_{ij}$ . By hypothesis there exists  $k \in J$  ( $k \neq j$ ) with  $\widehat{D}_{ik} < \widehat{d}_{ij}$ . Hence,  $d_{ij}^* > \widehat{D}_{ik} \geq d_{ik} = \|a_i - \bar{a}_k^*\|$ , so  $d_{ij}^* \neq \min_{j' \in J} \|a_i - \bar{a}_{j'}^*\|$  contradicting the optimality of the solution.
- (2) If  $x_{ij}^* = 0$ , then, there exists  $k \in J$ ,  $k \neq j$  such that  $d_{ij}^* \geq d_{ik}^* > \widehat{D}_{ij} \geq d_{ij}^*$ . Thus,  $x_{ij}^* = 1$ .
- (3) If applying 1, all the facilities except  $j$  must verify  $x_{ij'}^* = 0$ , then the unique choice for allocating  $i$  is  $j$ .

$\square$

As we will show in our computational experiments, the above strategies for fixing variables allows to fix an average of 80% of the binary variables in the test problems.

Using the above-described formulation, we implemented two different heuristic algorithms. Both algorithms will move through different feasible solutions in order to improve an initial given solution. This initial solution is constructed by either solving the standard DOMP problem with a set of weights based on the distances between centers of the neighborhoods (a convex combination of  $\widehat{D}_{ij}$  and  $\widehat{d}_{ij}$ ), or solving DOMP problem for simpler neighborhoods (as polyhedral neighborhoods) and polyhedral distances (which may require less computational effort than general  $\ell_\tau$ -norm based metrics or neighborhoods). Hence, we consider that an initial solution  $x^0 \in \mathcal{X}$  is known.

**4.1. Math-heuristic Algorithm 1.** Given a feasible solution  $\bar{x} \in \mathcal{X}$ , the first algorithm search, for each facility  $j_0$  in  $J$ , the best replacement by a facility in  $N \setminus J$ . Two different options are possible here. First, to construct the new set of open facilities  $J' = J \cup \{i\} \setminus \{j_0\}$  for each  $i \in N \setminus J$ , solve  $(\text{ALLOC}(J'))$  for such a  $J'$  and keep the best possible change for  $j_0$ . The second option is to solve a single mixed integer non linear programming problem which decides (through the binary variable  $\xi_i$ , whether the non-opened facility  $i$  is interchanged by  $j_0$  to obtain the best improvement.

$$\begin{aligned} (\text{BestRepl}(j)) \quad & \min \sum_{i \in N} u_i + \sum_{k \in N} v_k + \sum_{j \in J \setminus \{j_0\}} f_j + \sum_{i \in N \setminus J} f_i \xi_i, \\ \text{s.t.} \quad & u_i + v_k \geq \lambda_k z_i, \forall i, k \in N \setminus J, \end{aligned}$$

$$(4.2) \quad z_i \geq d_{ij} - \widehat{D}_{ij}(1 - x_{ij}), \forall i \in \{j_0\} \cup N \setminus J, j \in J \setminus \{j_0\},$$

$$(4.3) \quad z_i \geq d_{ij} - \widehat{D}_{ij}(2 - x_{ij} - \xi_j), \forall i \in \{j_0\} \cup N \setminus J, j \in J \setminus \{j_0\},$$

$$d_{ij} \geq \|a_i - \bar{a}_j\|, \forall i \in \{j_0\} \cup N \setminus J, j \neq j_0 \in J,$$

$$r_j \geq \|a_j - \bar{a}_j\|, \forall j \neq \{j_0\} \in N,$$

$$\begin{aligned} & \sum_{j \in N \setminus \{j_0\}} x_{ij} = 1, \forall i \in \{j_0\} \cup N \setminus J, \\ (4.4) \quad & x_{ij} \leq \xi_j, \forall j \in N \setminus J, \end{aligned}$$

$$(4.5) \quad \sum_{i \in N \setminus J} \xi_i = 1,$$

$$x_{ij}, \xi_i \in \{0, 1\}, \forall i \in \{j_0\} \cup N \setminus J, j \in N \setminus \{j_0\},$$

$$z_i \geq 0, \forall i \in N \setminus J.$$

$$(4.6)$$

Note that constraints (4.2) are the linearization of the bilinear terms as in the previous formulations, but obviating the facility that wants to be replaced ( $j_0$ ). For the candidates to replace  $j_0$ , constraints (4.3) assures that in case  $j$  is chosen for the replacement, and a customer  $i$  is allocated to  $j$ , then the travel cost for  $i$  is  $d_{ij}$  (the norm-based distance between  $i$  and  $j$  considered in the framework), otherwise, the constraint is redundant. With respect to the variables  $\xi$  that models the selection of the facility to be swapped with  $j_0$ , (4.4) ensures that unchosen facilities cannot serve any customer and (4.5) states that a single choice is possible. Although  $(\text{BestRepl}(j))$  is similar to  $(\text{DOMPN}_{BEP})$ , the number of binary variables it involves is slightly smaller, since it requires  $(n - p)n$  instead of  $n^2$ .

In our experiments, we have checked that solving  $(\text{BestRepl}(j))$  required more CPU time than solving the  $n - p$  problems in the form  $(\text{ALLOC}(J))$ , although for problems in which  $n - p \ll n$ , the *compact* formulation may consume less CPU time than loading and solving  $n - p$  problems in the shape of  $(\text{ALLOC}(J))$ .

In what follows we describe our math-heuristic procedure, whose pseudocode is shown in Algorithm 1. Given an initial set of  $p$  open facilities, it iterates by interchanging open facilities with other potential facilities trying to improve the best upper bound. At each iteration an open facility is selected to be replaced and the best replacement is chosen. After checking all the open facilities, if an improvement is found when compared to the best upper bound, the latest and the set of open facilities are updated. The procedure repeats the same scheme until a termination criterion is fulfilled. In our case, two stopping criteria are considered: maximum number of iterations and maximum number of iterations without improvement in the solution. In order to reduce the computation times required

for solvinf (ALLOC( $J$ )) or (BestRepl( $j$ )), we consider a randomized version of the algorithm in which instead of finding best replacements for all the open facilities, a random one is selected at that phase of the approach.

---

**Algorithm 1:** Math-Heuristic 1 for solving DOMP.

---

**Initialization:** Let  $\hat{J} \subset N$  with  $|\hat{J}| = p$  an initial set of open facilities and  $UB = c(\hat{J})$ .

```

while  $it < it_{max}$  do
  for  $j_0 \in \hat{J}$  do
    Find the best replacement for  $j$  (by solving (ALLOC( $J$ )) for
       $J = \hat{J} \cup \{i\} \setminus \{j\}$  or (BestRepl( $j$ )):  $c_{j_0} = c(J)$ .
    end
    if  $c_j < UB$  then
      Update  $UB = c_j$ 
       $\hat{J} = J$ 
      BREAK
    end
  end
  Increase  $it$ .
end

```

---

A key point of this algorithm is that an initial feasible solution (in the  $x$ -variables, i.e., in the discrete location-allocation decision variables) must be provided. We compute the solution of the DOMP problem (without neighborhoods) but using at costs between facilities  $i$  and  $j$  a convex combination of the lower and upper bounds  $\hat{d}_{ij}$  and  $\hat{D}_{ij}$ .

**4.2. Math-Heuristic Algorithm 2.** The second heuristic is based on alternating the location and allocation decisions. Initially, an standard DOMP problems by fixing  $\bar{a} = a$ , and precomputing the distances between the facilities. Once a solution is obtained, the optimal open facilities are kept and given as input to (ALLOC( $J$ )). Then, the variables  $\bar{a}$  are updated with the obtained solution and the process is repeated until stabilization. In order to escape from local optima, the scheme is applied again but forbidding the use of one of the facilities opened in the first stage. The process iterates until no improvements are found.

The pseudocode for this approach is shown in Algorithm 2.

---

**Algorithm 2:** Math-Heuristic 2 for solving DOMP.

---

**Initialization:**  $\bar{a} = a$

```

while  $|f_1 - f_2| > \varepsilon$  do
  • Solve DOMP for  $d_{ij} = \|a_i - \bar{a}_j\|$ . Update  $J = \{j \in N : x_{jj}^* = 1\}$  and its
    objective value  $f_1$ .
  • Solve (ALLOC( $J$ )) and update  $\bar{a}$ .
end
for  $j_0 \in J$  do
  Initialize  $\bar{a} = a$ .
  while  $|f_1 - f_2| > \varepsilon$  do
    • Solve DOMP for  $d_{ij} = \|a_i - \bar{a}_j\|$  forbidding opening  $j_0$ . Update
       $J = \{j \in N : x_{jj}^* = 1\}$  and its objective value  $f_1$ .
    • Solve (ALLOC( $J$ )) and update  $\bar{a}$  and its objective value  $f_2$ .
  end
end
end

```

---

In order to test the performance of the math-heuristic approaches, we have run some experiments over the real dataset instance of 2-dimensional coordinates (normalized longitude and latitude) of geographical centers of 49 states of the Unites

States (we exclude Alaska, Hawaii and those outside Northamerica). We considered as neighborhoods Euclidean disks with radii based on the areas of each state. For each state (indexed by  $j$ ), the area (in  $\text{km}^2$ ),  $A_j$ , was obtained and we construct the radius  $r_j^0 = \sqrt{\frac{A_j}{\pi}}$ . The coordinates and the disks built with this strategy are drawn in Figure 5. Then, three different scenarios were considered:

**S1:** :  $r_j = r_j^0$ , for  $j = 1, \dots, 49$ .

**S2:** :  $r_j = 2 \times r_j^0$ , for  $j = 1, \dots, 49$ .

**S3:** :  $r_j = 3 \times r_j^0$ , for  $j = 1, \dots, 49$ .

The data used for the experiments are available at [bit.ly/datasetUS](http://bit.ly/datasetUS).

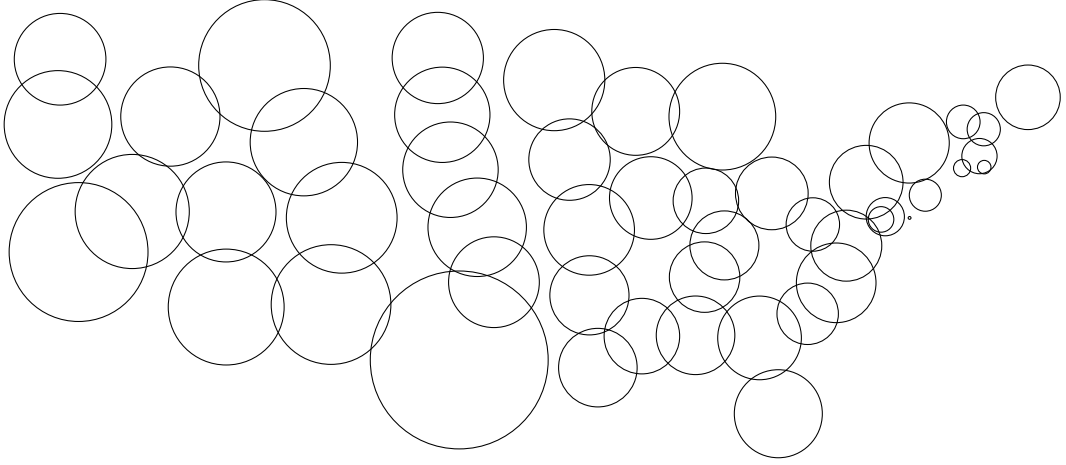


FIGURE 5. Basic neighborhoods for the 49 US states (radii  $r^0$ ).

We implemented in Gurobi under the C API the two math-heuristic approaches and we compare the running times and the best values obtained with these procedures and those obtained with the exact (DOMPN<sub>BEP</sub>) formulation (with a time limit of 1 hour). We solve the  $p$ -Median (M),  $p$ -Center (C),  $p$ -25-center (K) and  $p$ -Centdian (D) with  $p \in \{2, 5, 10\}$ . The results are reported in Table 13. In such a table, the first column indicates the scenario (1, 2 or 3), the second column (Pr.) shows the problem type and the third column indicates the number of facilities to be open,  $p$ . The values of the solutions obtained by using the different approaches as well as their CPU running times (in seconds) are reported:

- Initial solution obtained by solving the nominal DOMP problem and solving (ALLOC( $J$ )) for the obtained open facilities: **H0** and **t0**.
- Best solution obtained by the math-heuristic approach 1: **H1** and **t1**.
- Best solution obtained by the math-heuristic approach 2: **H2** and **t2**.
- Best solution obtained by exact formulation (DOMPN<sub>BEP</sub>) within the time limit: **BEP** and **tBEP**.

We also report in the 12th column (**%VarFixed**) the average number of binary variables fixed in the first heuristic, and the percentage deviations of the obtained solutions with respect to the best solution found with the exact formulation within the time limit: **G1**, **G2** and **G0** for the first heuristic, the second heuristic and the initial solution, respectively.

One can observe from the results that, the CPU times needed to run the math-heuristic approaches are much smaller than those needed to solve the DOMPN problem with the MINLP formulation. In those cases in which all the approaches

were able to solve the problem before the time limit of one hour, the highest deviations with respect to the optimal solutions were 15% for the first heuristic and 3.2% for the second one. In those cases in which the exact approach was not able to certify optimality in one hour, in most of the cases, some of the heuristic found in less than one hour a better solution for the DMPN problem. In the first heuristic, we apply the fixing variables strategy each time (ALLOC( $J$ )) is solved. The average percentage of binary variables that are fixed with this strategy, is at least 84% for scenario SC1, 75% for SC2 and 52% for SC3. Observe also that the initial solution based on fixing the open facilities to the solution of the DMP problem and then compute the location on the neighborhoods and the allocation of the customers according to these positions, is in some case far of being a close-to-optimal choice, with percentage deviations of 33% in some cases.

Note that the two math-heuristic approaches are very time consuming. The advantage of the two approaches is that they provided good quality solutions at the first iterations, which are competitive with the exact solutions (in terms of gap). One may not forget that both proposed approaches are based on solving mixed integer non linear programming problems which are known to be NP-hard.

In Figures 6 and 10 we show the best solutions obtained for the test problem for  $p = 5$  and  $p = 10$  under the center objective function for scenario SC1, the initials solutions for these problems are drawn in Figures 7 and 11, respectively. The reader can observe that *slight* modification of the coordinates of the potential facilities (through neighborhoods) may produce different location-allocation solutions.

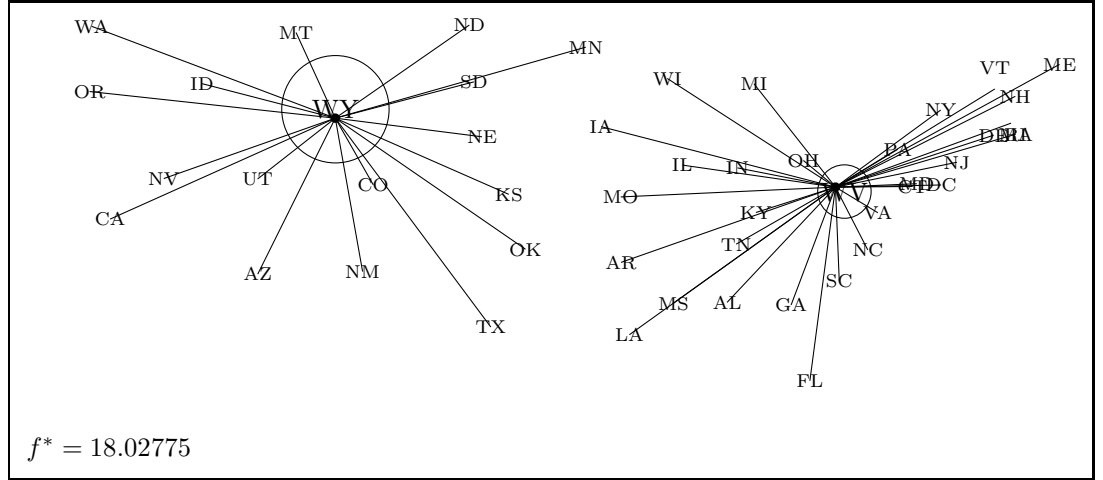


FIGURE 6. Solutions for 2-center problem under Scenario SC1 for the US data set.

## 5. CONCLUSIONS

A new version of the classical  $p$ -median problem is introduced in this paper which includes as particular cases the discrete and the continuous  $p$ -median and  $p$ -center problems. The problem considers that each facility can be located not only in the exact given position but in a neighborhood around it. Also, ordered median objective functions are modeled for the problem. Several mathematical programming formulations are proposed based on formulations for the discrete ordered median problem obtained from different sources. Two math-heuristic approaches for solving the problem are presented. Although the optimization problems needed to solve



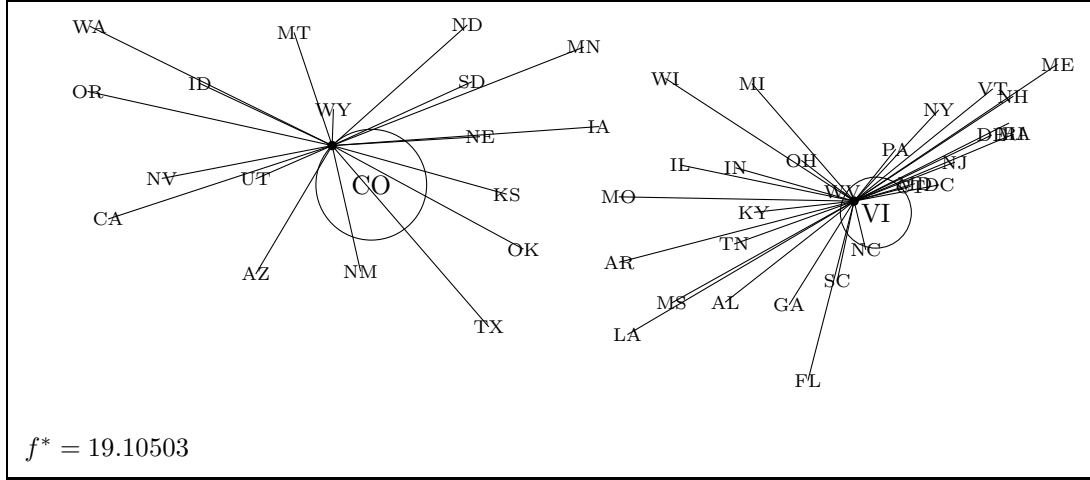


FIGURE 7. Initial solution for 2-center problem under Scenario SC1 for the US data set.

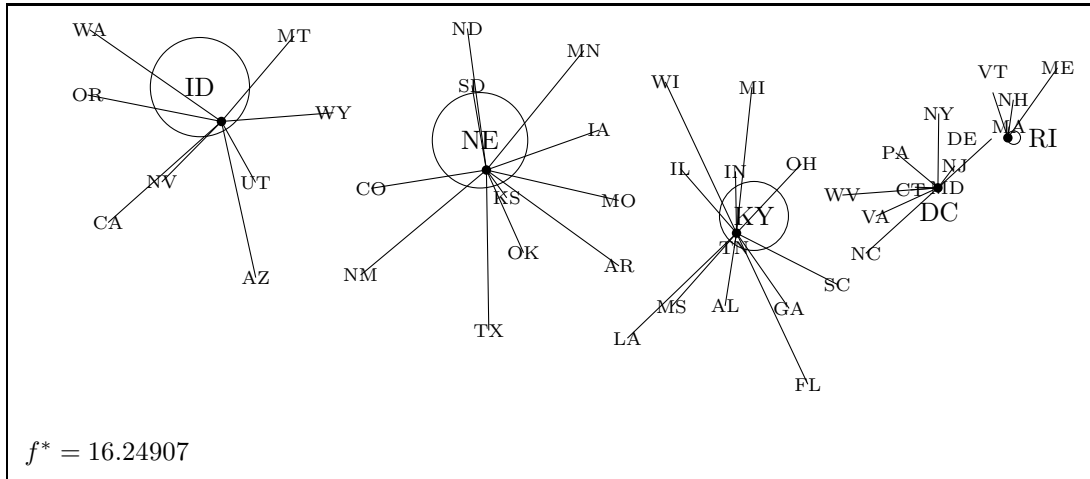


FIGURE 8. Solutions for 5-center problem under Scenario SC1 for the US data set.

are still NP-hard, the reduced dimension of them allows to provide good quality solution in more reasonable times.

Several extensions are possible within this new framework. The first is the development of decomposition approaches for solving DOMP. Lagrangean decomposition (relaxing the ordering constraints) combined with Benders decomposition (to *separate* the discrete and the continuous decisions) may produce exact solutions in better CPU times. On the other hand, although we analyze the ordered  $p$ -median problem, the results in this paper can be extended to other discrete location problems. For instance, capacitated [37] or multiperiod [1, 33] location problems can be embedded into the neighborhoods framework. Other interesting related problems which are left for further research is the consideration of location-routing problems with neighborhoods. That problem would involve not only the discrete facility location problem with neighborhoods but also the TSP with neighborhoods, then, the combination of the methods proposed in this paper with those provided in [18]

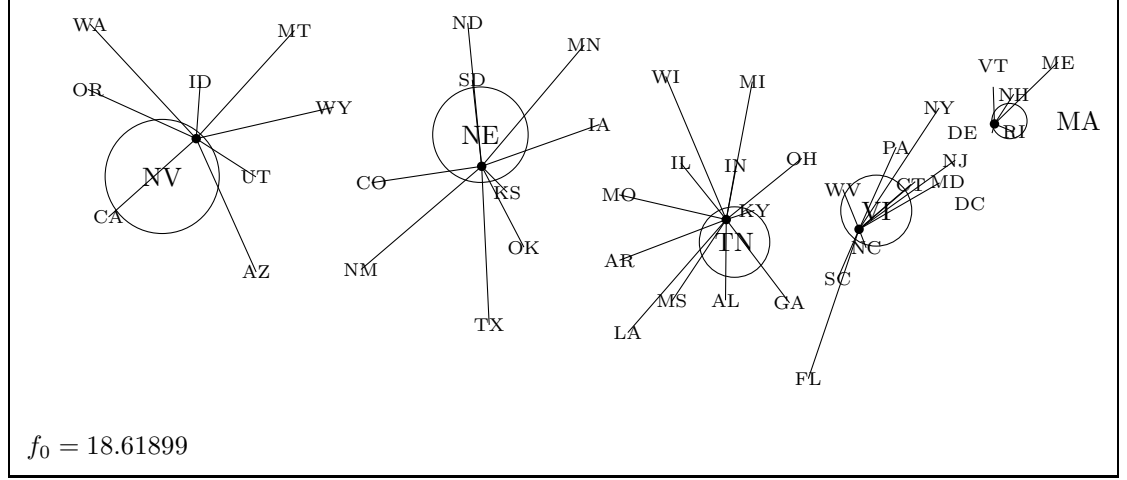


FIGURE 9. Initial Solutions for 5-center problem under Scenario SC1 for the US data set.

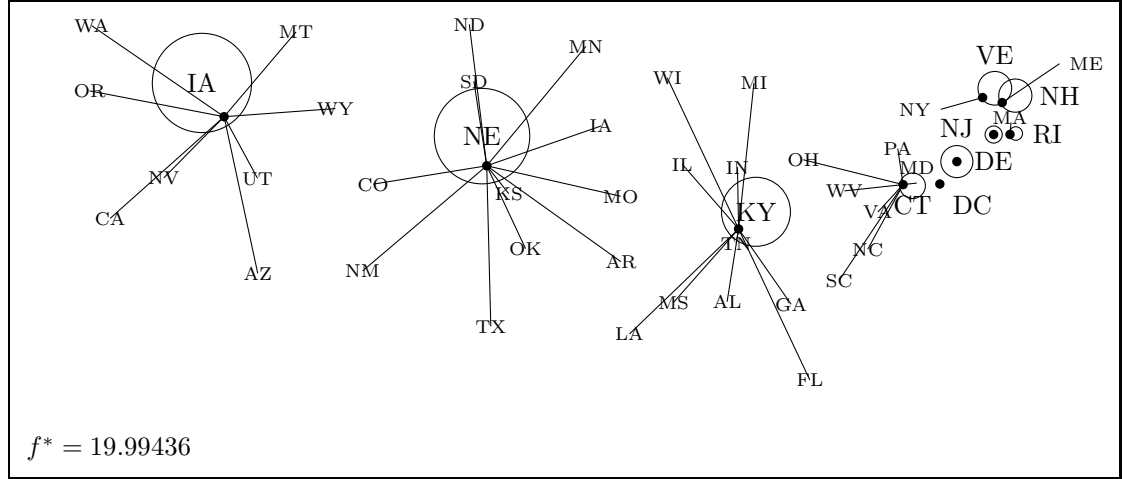


FIGURE 10. Solutions for 10-center problem under Scenario SC1 for the US data set.

may be applicable to the problem. Also, the case in which the neighborhood of each facility is the union of convex sets would be an interesting next step within this framework. In particular, it would model the case in which two facilities may belong to the same neighborhood. The extended MINLP formulations for such a problem will become disjunctive MINLP for which some techniques are available in the literature. Another approach that would extend the version introduced through this paper is the one in which  $k_j$  facilities are allowed to be located at the  $j$ -th neighborhood to allocate the demand points. In such a case, a nested multifacility  $p$ -median problem is considered for which more sophisticated strategies should be developed to solve even small-size instances.

#### ACKNOWLEDGEMENTS

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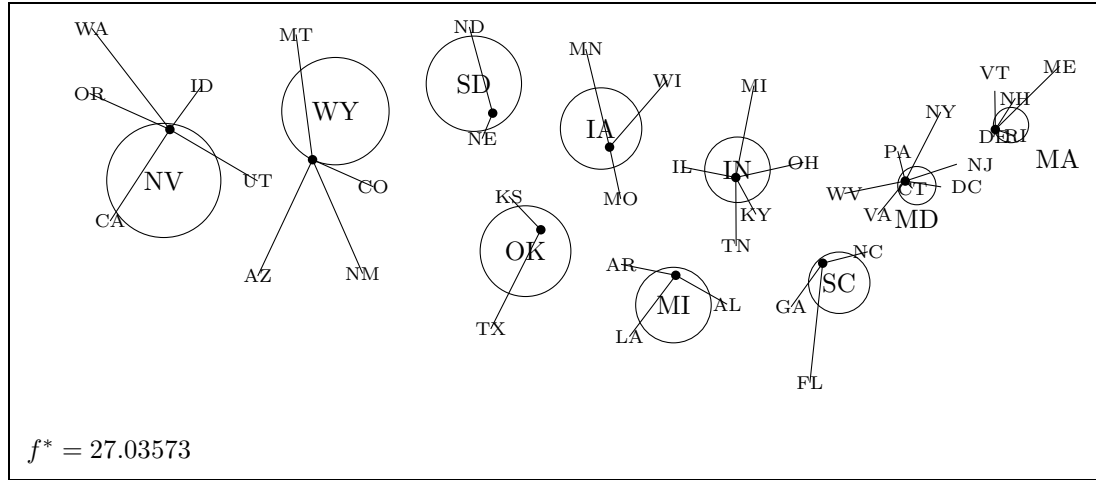


FIGURE 11. Solutions for 5-center problem under Scenario SC1 for the US data set.

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DPT. QUANT. METHODS FOR ECONOMICS & BUSINESS, UNIVERSIDAD DE GRANADA

n	Pr. p	Sc. 1			Sc. 2			Sc. 3			Sc. 4			
		IG <sub>3I</sub>	IG <sub>2I</sub>	IG <sub>BEP</sub>	IG <sub>3I</sub>	IG <sub>2I</sub>	IG <sub>BEP</sub>	IG <sub>3I</sub>	IG <sub>2I</sub>	IG <sub>BEP</sub>	IG <sub>3I</sub>	IG <sub>2I</sub>	IG <sub>BEP</sub>	
5	C	2	1608.97%	290.74%	424.35%	484.72%	397.56%	251.21%	722.70%	417.81%	265.11%	1075.49%	385.47%	298.45%
		3	1114.84%	402.55%	374.75%	280.33%	265.15%	199.95%	341.52%	293.49%	207.28%	574.26%	299.72%	219.29%
		4	561.87%	374.04%	279.84%	164.86%	164.06%	143.99%	188.17%	177.91%	142.37%	235.15%	201.77%	148.00%
	D	2	2660.19%	183.66%	313.40%	756.66%	619.86%	251.16%	1134.58%	645.19%	257.30%	1728.50%	551.98%	274.88%
		3	1453.95%	276.13%	273.13%	337.01%	318.63%	189.26%	412.61%	353.14%	193.75%	730.73%	364.39%	206.47%
		4	561.87%	370.48%	207.42%	164.86%	164.06%	131.61%	188.17%	177.58%	134.37%	235.15%	192.85%	143.62%
	K	2	2862.84%	284.17%	451.38%	798.33%	653.50%	313.97%	1207.24%	687.91%	321.16%	1834.69%	607.17%	348.73%
		3	1784.17%	436.59%	410.45%	387.21%	366.05%	237.50%	478.42%	409.37%	236.67%	876.73%	439.64%	253.66%
		4	561.87%	370.60%	228.75%	164.86%	164.06%	135.75%	188.17%	177.58%	134.80%	235.15%	193.59%	143.59%
	M	2	3682.06%	157.20%	290.41%	998.46%	817.51%	252.04%	1520.03%	864.08%	259.07%	2336.83%	712.17%	271.57%
		3	1784.18%	237.42%	246.14%	387.21%	366.05%	186.95%	478.42%	409.37%	191.46%	876.73%	437.43%	206.24%
		4	561.87%	370.48%	186.88%	164.86%	164.06%	127.67%	188.17%	177.58%	133.51%	235.15%	192.26%	143.35%
6	C	2	1414.69%	397.36%	416.00%	830.65%	534.07%	327.10%	285.55%	279.57%	210.17%	684.81%	446.42%	264.40%
		3	805.95%	436.89%	345.30%	376.67%	356.19%	248.59%	192.34%	191.33%	161.20%	290.50%	273.33%	192.71%
		4	437.61%	378.46%	283.76%	243.48%	239.51%	197.49%	149.53%	149.31%	135.09%	192.28%	189.28%	154.45%
	D	2	2724.03%	351.85%	364.51%	1541.42%	912.68%	327.55%	481.12%	470.91%	255.34%	1223.26%	798.19%	291.90%
		3	1226.43%	531.88%	295.38%	548.49%	513.96%	244.95%	246.09%	244.76%	174.60%	403.58%	379.49%	207.63%
		4	525.64%	444.26%	234.84%	276.95%	272.41%	181.57%	157.98%	157.75%	132.43%	207.91%	204.58%	149.13%
	K	2	219.53%	212.62%	152.79%	149.54%	149.12%	126.27%	112.58%	112.58%	107.11%	127.59%	127.59%	113.72%
		3	3427.24%	488.56%	489.16%	1905.92%	1127.27%	408.33%	573.16%	560.89%	309.29%	1482.69%	966.44%	342.05%
		4	1633.91%	719.15%	409.43%	709.02%	664.75%	310.18%	294.43%	292.82%	209.15%	502.64%	472.13%	245.57%
	M	2	610.03%	514.27%	280.04%	309.22%	304.13%	199.51%	165.33%	165.09%	138.07%	223.16%	219.52%	154.69%
		3	219.53%	212.62%	153.80%	149.54%	149.12%	125.68%	112.58%	112.58%	107.28%	127.59%	127.59%	112.75%
		4	3947.92%	336.32%	353.87%	2188.80%	1238.64%	329.19%	651.66%	637.71%	280.05%	1710.07%	1115.91%	305.93%
7	C	2	1633.91%	639.80%	287.21%	709.02%	663.59%	247.91%	294.43%	292.82%	186.32%	502.64%	472.13%	217.07%
		3	610.03%	504.30%	220.46%	309.22%	304.13%	176.61%	165.33%	165.09%	132.11%	223.16%	219.52%	148.75%
		4	219.53%	212.62%	141.50%	149.54%	149.12%	121.71%	112.58%	112.58%	106.13%	127.59%	127.59%	112.54%
	D	2	2026.94%	402.20%	495.54%	2070.62%	463.81%	460.28%	318.82%	298.18%	221.70%	223.83%	222.85%	187.59%
		3	1344.22%	444.39%	407.27%	1413.83%	470.10%	384.66%	201.81%	201.48%	169.15%	159.55%	159.47%	145.20%
		4	715.19%	375.44%	299.94%	660.82%	363.51%	279.13%	147.88%	147.87%	135.02%	126.57%	126.57%	119.83%
	K	2	368.63%	294.72%	230.67%	378.57%	280.42%	214.82%	123.67%	123.67%	117.05%	111.72%	111.72%	108.97%
		3	4512.17%	375.54%	442.33%	4619.85%	556.15%	434.10%	614.48%	574.45%	302.52%	398.64%	396.87%	263.95%
		4	2389.81%	461.55%	357.65%	2507.84%	700.03%	345.15%	290.19%	289.71%	201.55%	211.12%	211.02%	171.98%
	M	2	1107.01%	536.10%	277.65%	1012.89%	523.80%	266.92%	176.65%	176.64%	146.02%	141.36%	141.36%	127.30%
		3	467.96%	370.81%	212.80%	464.34%	336.69%	207.53%	131.39%	131.39%	119.61%	115.24%	115.24%	110.56%
		4	5076.81%	516.67%	588.48%	5160.01%	711.94%	569.80%	672.96%	629.05%	350.48%	434.71%	432.76%	297.73%
8	C	2	2865.50%	623.89%	481.56%	3006.06%	865.75%	447.50%	333.34%	332.78%	237.12%	235.68%	235.57%	195.32%
		3	1484.33%	726.62%	389.96%	1356.84%	702.00%	353.60%	202.02%	202.00%	167.05%	153.89%	153.89%	139.33%
		4	565.12%	447.20%	258.83%	549.49%	397.76%	232.06%	138.12%	138.12%	125.11%	118.15%	118.15%	113.32%
	D	2	6784.48%	382.93%	429.35%	6890.73%	658.82%	425.31%	868.26%	811.80%	340.98%	546.72%	544.29%	307.82%
		3	3368.96%	502.15%	345.07%	3542.95%	926.90%	338.13%	366.08%	365.46%	220.65%	254.62%	254.50%	190.50%
		4	1484.33%	685.71%	273.66%	1356.84%	700.54%	271.25%	202.02%	202.00%	154.97%	153.89%	153.89%	133.71%
	K	2	565.12%	447.20%	211.53%	549.49%	397.76%	209.00%	138.12%	138.12%	122.42%	118.15%	118.15%	112.07%
		3	2569.93%	345.91%	567.47%	1219.97%	670.70%	386.78%	673.47%	556.75%	316.06%	480.96%	437.32%	278.29%
		4	1440.16%	387.40%	428.75%	525.32%	444.60%	284.62%	296.35%	283.84%	209.19%	277.74%	265.54%	191.62%
	M	2	882.43%	412.37%	363.84%	328.19%	307.67%	227.10%	204.05%	200.77%	164.29%	211.24%	205.66%	159.96%
		3	572.77%	381.55%	304.96%	239.78%	231.04%	185.65%	163.24%	162.93%	142.31%	173.80%	173.39%	144.71%
		4	6564.95%	244.08%	506.02%	2998.34%	1629.26%	436.21%	1550.52%	1280.94%	392.02%	1090.27%	990.33%	376.70%
9	C	2	3152.37%	321.19%	396.59%	1044.30%	881.04%	324.76%	549.80%	526.35%	274.52%	512.63%	489.65%	254.82%
		3	1637.03%	501.17%	343.67%	535.98%	502.14%	257.11%	307.12%	302.14%	203.11%	322.28%	313.48%	198.19%
		4	862.11%	553.64%	286.41%	318.46%	306.81%	199.40%	194.28%	193.90%	152.21%	214.39%	213.87%	158.41%
	D	2	8094.37%	334.91%	647.38%	3665.62%	1991.61%	529.14%	1862.21%	1538.42%	458.92%	1306.65%	1186.64%	454.79%
		3	4331.56%	479.14%	563.72%	1384.28%	1167.35%	410.01%	709.68%	679.39%	340.23%	665.00%	635.14%	321.42%
		4	2373.15%	748.79%	504.40%	732.47%	686.23%	335.07%	394.15%	387.70%	249.56%	421.08%	409.45%	244.85%
	K	2	1137.75%	730.66%	371.69%	391.55%	377.23%	233.99%	223.48%	223.04%	167.93%	251.16%	250.53%	174.30%
		3	10204.01%	219.87%	487.63%	4603.87%	2502.28%	456.76%	2304.67%	1903.09%	420.47%	1637.28%	1486.43%	417.64%
		4	4824.17%	308.50%	395.64%	1540.79%	1299.56%	344.66%	772.32%	739.24%	305.85%	731.43%	698.38%	293.75%
	M	2	2373.15%	576.26%	348.85%	732.48%	686.24%	277.48%	394.15%	387.70%	226.65%	421.08%	409.45%	223.55%
		3	1137.75%	712.98%	286.72%	391.55%	377.23%	210.08%	223.48%	223.04%	161.33%	251.16%	250.53%	169.91%
		4	3459.90%	377.51%	631.63%	3134.09%	621.57%	524.55%	847.92%	475.19%	304.18%	1430.15%	495.87%	330.16%
10	C	2	2228.60%	451.75%	518.26%	1732.07%	533.83%	420.77%	513.01%	355.83%	248.06%	713.37%	397.65%	274.19%
		3	1412.08%	433.41%	400.55%	956.68%	399.95%	320.68%	266.61%	264.30%	206.08%	334.14%	283.06%	211.30%
		4	948.45%	388.12%	342.93%	619.94%	336.71%	267.47%	184.85%	184.61%	157.92%	206.98%	205.03%	170.06%
	D	2	9678.86%	270.53%	565.80%	8617.78%	1191.36%	533.40%	2159.10%	1209.79%	432.95%	3703.93%	1282.79%	442.37%
		3	5449.40%	395.33%	479.39%	4167.76%	1202.14%	443.41%	1062.58%	736.79%	332.14%	1550.84%	862.58%	351.24%
		4	3173.31%	601.73%	403.46%	2055.09%	816.14%	356.35%	452.63%	448.66%	257.09%	599.04%	506.87%	268.75%
	K	2	1785.40%	647.35%	344.14%	1120.24%	588.01%	301.00%	260.74%	260.39%	188.44%	295.08%	292.28%	200.34%
		3	11079.40%	380.73%	737.34%	9729.04%	1423.70%	671.60%	2394.38%	1341.74%	505.26%	4132.17%	1431.55%	507.45%
		4	6601.62%	566.36%	651.79%	4961.11%	1429.65%	556.01%	1240.55%	860.08%	380.38%	1837.16%	1021.72%	402.29%
	M	2	4248.14%	872.42%	564.23%	2713.14%	1079.49%	453.94%	562.54%	557.60%	299.45%	750.98%	635.30%	312.07%
		3	2613.89%	955.01%	489.69%	1597.55%	839.26%	383.35%	329.18%	328.74%	226.08%	372.65%	369.09%	233.98%
		4	15558.24%	250.00%	555.95%	13755.22%	1646.59%	540.24%	3336.35%	1869.16%	480.72%	5672.27%	1964.62%	480.96%
11	C	2	8513.74%	383.89%	474.50%	6428.17%	1848.35%	451.57%	1550.04%	1074.48%	371.02%	2272.03%	1263.65%	382.96%
		3	4861.61%	725.95%	410.43%	3108.23%	1235.52%	376.81%	618.82%	613.39%	285.29%	826.39%	699.22%	296.83%
		4	2613.89%	915.56%	354.39%	1597.55%	838.54%	321.28%	329.18%	328.74%	208.35%	372.65%	369.09%	219.68%
	D	2	3204.62%	442.38%	631.91%	3131.74%	605.50%	569.00%	998.10%	644.37%	383.48%	1796.27%	538.67%	371.76%
		3	1942.55%	483.36%	505.83%	2013.89%	561.03%	466.41%	582.44%	444.69%	296.41%	874.0		

$n$	$p$	Scenario 1	Scenario 2	Scenario 3	Scenario 4
5	2	100.00%	0.00%	0.00%	0.00%
	3	50.00%	0.00%	0.00%	0.00%
	4	0.00%	0.00%	0.00%	0.00%
6	2	52.50%	0.00%	0.00%	0.00%
	3	20.00%	0.00%	0.00%	0.00%
	4	0.00%	0.00%	0.00%	0.00%
	5	0.00%	0.00%	0.00%	5.00%
7	2	80.00%	52.50%	0.00%	0.00%
	3	22.50%	0.00%	0.00%	0.00%
	4	0.00%	0.00%	0.00%	0.00%
	5	0.00%	0.00%	0.00%	0.00%
8	2	100.00%	0.00%	0.00%	0.00%
	3	80.00%	0.00%	0.00%	0.00%
	4	10.00%	0.00%	0.00%	0.00%
	5	0.00%	0.00%	0.00%	0.00%
9	2	100.00%	2.50%	0.00%	0.00%
	3	90.00%	0.00%	0.00%	0.00%
	4	15.00%	0.00%	0.00%	0.00%
	5	0.00%	0.00%	0.00%	0.00%
10	2	90.00%	32.50%	0.00%	0.00%
	3	50.00%	0.00%	0.00%	0.00%
	4	5.00%	0.00%	0.00%	0.00%
	5	0.00%	0.00%	0.00%	0.00%

TABLE 3. Percentage of instances in which  $z_{2I}^R \geq z_{BEP}^R$ .

$n$	Pr.	$p$	(DOMPN <sub>3I</sub> )			(DOMPN <sub>2I</sub> )			(DOMPN <sub>OT</sub> )			(DOMPN <sub>BEP</sub> )		
			time	#nodes	%gapR	time	#nodes	%gapR	time	#nodes	%gapR	time	#nodes	%gapR
10	C	2	217.1854	118505.20	96.7083%	26.7283	51583.60	85.0086%	0.3911	79.60	32.9740%	0.2140	56.20	33.8840%
		3	766.0910	391417.00	93.8452%	139.8175	268504.80	84.0571%	0.3656	159.40	38.9007%	0.2648	103.20	36.9723%
		4	3375.4294	872945.80	88.8183%	1088.9767	1624482.00	78.9859%	0.3142	180.20	39.7755%	0.1915	54.80	36.3401%
		5	3527.0822	1109172.20	83.9769%	2225.2950	4064066.40	75.1986%	0.2927	217.80	45.2287%	0.1810	87.40	39.0692%
		5	78.4928	17176.80	98.8950%	20.8210	24835.40	86.8206%	0.3507	55.40	9.1685%	0.2585	34.00	4.4884%
	D	3	142.1779	15663.80	97.5435%	21.5641	30214.20	93.4823%	0.3462	6.60	10.0124%	0.2224	0.20	0.0000%
		4	68.8762	5336.60	95.0042%	22.7586	40081.00	90.4015%	0.3358	4.80	8.8556%	0.1468	0.20	0.0000%
		5	79.8333	10692.00	91.4336%	147.9079	304842.40	86.5397%	0.4274	45.00	15.0702%	0.2659	18.60	8.2553%
		2	291.0093	136621.00	99.1393%	113.0029	184032.00	90.4703%	0.2781	69.80	26.1355%	0.3371	88.40	23.2247%
		3	2239.4365	961277.00	98.2028%	457.9558	580536.00	95.2467%	0.3154	135.00	31.1459%	0.3047	103.40	26.9918%
	K	4	3079.6584	745095.80	96.5506%	1538.1743	1946764.20	93.3791%	0.3606	169.20	37.9192%	0.3085	123.40	32.7199%
		5	613.0526	246909.80	94.0944%	4081.5631	3501105.00	90.7190%	0.3847	259.40	42.6846%	0.3364	183.20	36.2925%
		2	45.9836	9877.60	99.3231%	7.3247	7234.40	86.9328%	0.3904	26.60	8.1909%	0.2603	5.20	2.2115%
		3	35.9483	6140.80	98.4262%	10.4362	6924.00	95.8277%	0.2331	6.80	2.4264%	0.1590	1.60	0.7074%
		4	32.0741	6413.60	96.7557%	3.4491	4077.80	93.7674%	0.2433	26.60	7.8241%	0.1494	0.20	0.0000%
	M	5	24.7478	4168.00	94.0944%	4.4173	4489.20	90.7190%	0.2733	9.20	5.3678%	0.1383	0.40	0.0000%

TABLE 4. Average results of comparing the formulations for the planar case (Scenario 1).

$n$	Pr.	$p$	(DOMPN <sub>3I</sub> )			(DOMPN <sub>2I</sub> )			(DOMPN <sub>OT</sub> )			(DOMPN <sub>BEP</sub> )		
			time	#nodes	%gapR	time	#nodes	%gapR	time	#nodes	%gapR	time	#nodes	%gapR
10	C	2	185.7869	243222.20	97.0569%	35.6459	160015.80	87.1104%	0.1844	100.40	33.2089%	0.1200	62.20	33.5438%
		3	3368.9779	2751134.40	94.7102%	157.5736	564670.20	82.5477%	0.1781	203.40	38.3630%	0.1509	165.00	31.6213%
		4	3056.5091	152278.20	90.9189%	557.2536	1636689.80	77.6323%	0.1800	255.40	38.0286%	0.1232	114.20	28.8268%
	D	5	312.5842	319355.20	82.9186%	334.0322	1082684.60	73.4912%	0.1764	234.00	38.9988%	0.1221	140.20	26.2644%
		2	40.0176	20172.60	98.9975%	8.6393	23279.00	95.4316%	0.1272	38.00	13.7618%	0.1402	31.80	8.5492%
		3	76.9185	30308.20	97.8742%	11.7149	34052.20	92.7448%	0.1276	22.00	10.8993%	0.0883	12.40	7.5741%
		4	93.4193	15935.20	95.8452%	10.5942	39267.20	89.5056%	0.1363	11.80	7.9720%	0.1298	0.60	4.0184%
	K	5	66.5258	15954.20	90.7396%	14.5289	70825.60	85.3482%	0.1643	40.00	14.5225%	0.1254	0.60	2.1163%
		2	257.2492	239132.80	99.2092%	94.2836	289128.40	96.4121%	0.1419	101.80	27.7109%	0.1646	105.80	25.6993%
		3	1804.5051	1410405.00	98.4371%	320.7443	836508.40	94.7125%	0.1340	116.00	32.2083%	0.2069	177.80	29.3818%
	M	4	2053.0221	1469281.00	97.0779%	724.5832	1705800.00	92.6398%	0.1705	177.00	37.5458%	0.1945	185.00	34.1132%
		5	833.5048	774058.20	93.5253%	3164.0954	6317876.00	89.7517%	0.1561	190.80	38.5922%	0.1776	245.80	34.5138%
		2	37.7167	18920.00	99.3798%	5.0132	9075.20	97.1696%	0.1597	42.60	11.3291%	0.1465	28.60	7.2125%
		3	27.1995	11529.40	98.6339%	4.6558	7610.40	95.3531%	0.1701	4.80	4.2813%	0.0936	1.00	1.7813%
		4	24.2447	11298.60	97.2586%	4.8601	12168.00	93.0817%	0.1911	59.00	4.1480%	0.1178	1.00	0.2326%
		5	15.0723	5040.80	93.5253%	2.4553	5522.40	89.7517%	0.1980	46.40	8.4654%	0.1171	5.60	3.3069%

TABLE 5. Average results of comparing the formulations for the planar case (Scenario 2).

$n$	Pr.	$p$	(DOMPN <sub>3I</sub> )			(DOMPN <sub>2I</sub> )			(DOMPN <sub>OT</sub> )			(DOMPN <sub>BEP</sub> )		
			time	#nodes	%gapR	time	#nodes	%gapR	time	#nodes	%gapR	time	#nodes	%gapR
10	C	2	173.1471	247260.00	88.9275%	10.5746	31894.00	83.4021%	0.1364	52.40	38.7733%	0.0956	59.80	29.8302%
		3	1319.3256	1565352.60	79.4324%	87.5368	246224.40	74.0041%	0.1263	97.00	39.1824%	0.0957	56.60	24.9413%
		4	1143.9317	996110.00	65.6549%	128.5771	429914.60	62.6941%	0.1390	117.60	34.1734%	0.0939	47.40	18.4669%
	D	5	1035.3915	1506863.00	53.4857%	190.3161	751796.20	52.2358%	0.1734	206.40	32.2652%	0.1355	150.00	17.8483%
		2	48.2205	36372.20	96.0620%	4.1220	9171.00	94.0895%	0.1307	62.20	33.7972%	0.0925	42.60	15.0431%
		3	36.7032	20929.40	90.8633%	3.8903	10671.20	88.4309%	0.1203	60.20	37.0115%	0.1228	18.80	8.6888%
		4	340.4550	506451.80	81.7723%	4.2877	11882.80	80.1996%	0.1019	74.20	40.7362%	0.1280	3.80	3.0051%
	K	5	83.8378	172707.40	69.9151%	5.6643	23412.20	69.0827%	0.1263	88.20	35.3413%	0.1596	22.20	4.1523%
		2	558.2910	466004.40	96.8478%	131.9660	270282.20	95.2668%	0.1165	49.80	41.4985%	0.1222	42.40	27.5901%
		3	653.9674	510418.80	93.1742%	285.0070	572239.00	91.3581%	0.1374	101.80	48.2976%	0.1464	91.60	29.6278%
	M	4	309.6418	307686.80	86.8082%	1573.0557	2854817.00	85.6934%	0.1553	163.60	54.6024%	0.1358	105.60	31.4726%
		5	248.3248	247774.40	77.5109%	3732.5863	1571394.80	76.8801%	0.1737	224.80	55.8336%	0.1576	147.20	28.0373%
		2	50.2046	37663.80	97.5365%	3.9525	6822.60	96.2966%	0.1594	80.40	37.3292%	0.1123	56.00	16.1239%
		3	35.6821	21078.00	93.9312%	3.2528	4576.40	92.3103%	0.1065	59.80	41.4257%	0.1497	17.40	9.7970%
		4	22.9353	16293.00	87.3302%	1.9441	3479.40	86.2419%	0.1391	58.80	48.3440%	0.1359	3.00	5.7048%
		5	167.9641	230957.40	77.5109%	1.5040	1669.40	76.8801%	0.1385	93.00	43.8420%	0.1425	11.60	3.9405%

TABLE 6. Average results of comparing the formulations for the planar case (Scenario 3).

$n$	Pr.	$p$	(DOMPN <sub>3I</sub> )			(DOMPN <sub>2I</sub> )			(DOMPN <sub>OT</sub> )			(DOMPN <sub>BEP</sub> )		
			time	#nodes	%gapR	time	#nodes	%gapR	time	#nodes	%gapR	time	#nodes	%gapR
10	C	2	203.7713	245152.40	94.3110%	49.6031	127865.40	82.6978%	0.1394	139.00	34.4855%	0.1204	121.20	32.5743%
		3	1201.0587	1114765.20	86.8838%	529.5429	728381.60	76.3117%	0.1668	249.80	33.4565%	0.1045	82.40	26.9251%
		4	323.9521	498328.20	73.2863%	353.4516	676140.80	64.9752%	0.1427	126.20	18.4783%	0.0776	63.40	8.6506%
	D	5	1542.5574	110782.40	62.2897%	54.4626	151331.20	56.1496%	0.1310	217.40	23.0933%	0.0701	51.40	6.0516%
		2	423.3462	210911.80	97.9492%	13.3245	29156.00	93.4944%	0.1863	166.80	32.2787%	0.1732	174.80	29.1929%
		3	396.2104	211169.80	94.3114%	24.8441	50853.00	89.2074%	0.2062	241.80	32.7270%	0.2012	243.20	28.7333%
	K	4	213.0761	121462.00	86.6103%	18.5905	45899.00	81.7691%	0.2003	277.40	26.0344%	0.1624	244.20	21.9511%
		5	65.4929	49704.20	76.5250%	20.7402	54514.80	72.0584%	0.1675	160.00	18.8068%	0.1403	168.20	14.1709%
	M	2	700.3207	457647.40	98.3403%	349.4380	456972.60	94.7327%	0.1623	158.60	40.9255%	0.1756	136.40	38.2781%
		3	2865.2473	1667538.20	95.5906%	643.4857	870765.40	91.6433%	0.2266	296.80	40.9274%	0.2357	248.40	37.7500%
		4	2825.2499	736735.60	90.0239%	3361.0209	4465105.80	86.4185%	0.2579	404.80	38.8505%	0.2155	304.20	35.8343%
		5	1753.1540	257832.20	82.1222%	5823.2001	2160439.20	78.7083%	0.2332	372.80	32.6063%	0.1951	350.80	29.2638%
		2	576.7956	281412.80	98.6691%	6.9595	12096.60	95.7805%	0.1738	150.40	38.4161%	0.1602	133.60	33.2328%
		3	699.1064	296994.00	96.1984%	4.8967	7174.60	92.7993%	0.2157	313.40	41.0327%	0.1721	272.60	35.2902%
		4	170.9246	109314.00	90.5354%	6.6562	12752.60	87.1195%	0.1855	306.20	36.7044%	0.1737	280.60	30.0438%
		5	138.5395	102782.60	82.1222%	5.7353	13638.80	78.7085%	0.1567	209.00	29.0927%	0.1979	433.00	22.6414%

TABLE 7. Average results of comparing the formulations for the planar case (Scenario 4).



$n$	Pr.	$p$	(DOMP $N_{3I}$ )			(DOMP $N_{2I}$ )			(DOMP $N_{OT}$ )			(DOMP $N_{BEP}$ )		
			time	#nodes	%gapR	time	#nodes	%gapR	time	#nodes	%gapR	time	#nodes	%gapR
10	C	2	473.8436	368360.80	97.2242%	35.7374	52967.00	74.3749%	0.2437	82.20	24.0332%	0.1483	79.40	25.1746%
		3	5763.6494	845075.80	95.5936%	1675.2136	294038.40	79.7544%	0.2433	157.20	33.2582%	0.1478	126.60	35.4296%
		4	5430.2825	799358.60	92.7955%	4792.3017	494225.00	80.4389%	0.2588	216.80	38.5371%	0.1506	109.00	39.5682%
		5	3842.6682	611415.40	89.1261%	4801.6213	2054248.20	78.2513%	0.2344	203.20	41.7958%	0.1395	147.00	41.3691%
		2	61.9117	18930.00	99.1915%	88.5497	43946.00	68.9708%	0.1256	6.60	2.9946%	0.1530	2.60	0.9130%
	D	3	371.3063	47028.60	98.4949%	199.2775	68335.00	80.3328%	0.1774	29.60	6.2656%	0.1363	31.80	3.9603%
		4	132.8928	41835.00	97.1839%	142.3535	109411.80	87.9226%	0.1911	34.60	8.4188%	0.1284	16.60	5.9333%
		5	128.0219	28957.00	95.1406%	192.7043	260491.80	90.0602%	0.2404	79.80	8.1540%	0.1804	29.20	4.7258%
		2	511.7714	225792.80	99.3340%	181.7884	222769.20	76.4936%	0.2218	138.20	19.7114%	0.2186	146.80	17.5264%
		3	4807.0837	995386.60	98.8716%	4377.3969	1019791.00	86.4455%	0.2273	176.00	31.5811%	0.2605	235.00	29.1426%
	K	4	5913.2958	116629.00	98.0456%	4861.3838	2118727.80	92.0771%	0.2666	261.60	39.2123%	0.2731	226.00	36.3447%
		5	4950.3982	324797.00	96.8525%	6245.9024	2442139.80	93.5781%	0.2064	224.80	45.2093%	0.2259	308.80	41.7789%
	M	2	57.4718	18665.20	99.5232%	14.1877	13224.00	67.9183%	0.2081	32.40	3.8501%	0.1489	4.80	0.9935%
		3	98.5955	24726.80	99.0766%	75.0224	14259.60	80.2555%	0.1498	19.20	2.3703%	0.1140	36.20	1.6010%
		4	42.0597	18444.00	98.2292%	9.5997	11981.00	89.8295%	0.1657	14.20	2.5842%	0.1224	3.00	1.1491%
		5	27.4122	15402.40	96.8525%	4.9887	9265.40	93.4312%	0.1481	3.20	2.7249%	0.1410	3.80	0.2195%

TABLE 8. Average results of comparing the formulations for the 3-dimensional case (Scenario 1).

$n$	Pr.	$p$	(DOMP $N_{3I}$ )			(DOMP $N_{2I}$ )			(DOMP $N_{OT}$ )			(DOMP $N_{BEP}$ )		
			time	#nodes	%gapR	time	#nodes	%gapR	time	#nodes	%gapR	time	#nodes	%gapR
10	C	2	698.7996	582674.60	97.0546%	52.2661	98390.60	84.3896%	0.1492	50.60	23.9449%	0.1278	67.60	25.0377%
		3	6236.3041	1034617.60	95.6016%	1624.7183	1452726.00	84.2460%	0.1759	123.20	32.2179%	0.1186	56.60	33.3922%
		4	5647.4030	696189.80	93.4638%	3311.3159	3658955.80	82.0392%	0.2032	176.40	36.3640%	0.1679	130.20	35.8261%
		5	5778.9274	20217.40	88.7362%	5513.0253	2033249.20	79.1087%	0.2011	179.80	38.0032%	0.1094	56.40	35.6393%
		2	26.9657	13871.00	99.1384%	10.6378	21167.80	91.2645%	0.1338	19.00	7.3001%	0.1627	10.60	5.6396%
	D	3	26.0877	16012.40	98.4796%	7.3006	14069.60	94.3503%	0.1226	28.40	7.3883%	0.1428	39.00	5.1628%
		4	45.6987	10844.00	97.3814%	18.5005	30920.40	92.6049%	0.1253	55.60	8.7498%	0.1341	38.40	7.0831%
		5	23.1696	4170.60	94.7557%	11.4018	40594.20	90.0580%	0.1573	88.00	9.5335%	0.1662	13.80	4.3752%
		2	1785.2056	167030.20	99.2846%	255.9488	299773.60	93.0701%	0.1289	41.20	20.3732%	0.1561	40.80	18.8201%
		3	3734.8218	368152.20	98.8577%	2184.1479	792684.60	95.7859%	0.1557	86.60	32.3946%	0.1604	78.20	30.6603%
	K	4	2497.7127	540841.00	98.1554%	4728.8366	1716593.40	94.8174%	0.1697	137.40	38.3435%	0.1663	121.80	36.3419%
		5	3099.2334	160480.80	96.5693%	6860.2279	3165215.00	93.4937%	0.1856	188.20	43.5332%	0.1830	163.80	41.2300%
	M	2	22.8391	12370.60	99.4877%	5.4637	7523.80	92.4120%	0.1436	13.40	6.9343%	0.1599	8.40	2.5046%
		3	25.4442	11016.80	99.0615%	4.8358	7306.80	96.4866%	0.1015	19.80	6.1189%	0.1604	17.20	3.9214%
		4	16.4418	7086.40	98.3319%	2.7428	4104.80	95.2955%	0.1402	19.20	6.0424%	0.1330	18.20	3.7025%
		5	12.7005	2887.40	96.5693%	2.2385	2401.80	93.4937%	0.1457	18.40	1.5623%	0.0997	3.00	0.7059%

TABLE 9. Average results of comparing the formulations for the 3-dimensional case (Scenario 2).

$n$	Pr.	$p$	(DOMP $N_{3I}$ )			(DOMP $N_{2I}$ )			(DOMP $N_{OT}$ )			(DOMP $N_{BEP}$ )		
			time	#nodes	%gapR	time	#nodes	%gapR	time	#nodes	%gapR	time	#nodes	%gapR
10	C	2	420.0330	251532.60	90.6794%	44.8147	67897.80	84.8324%	0.2064	75.00	28.9238%	0.1230	63.00	22.8087%
		3	4166.5666	629476.20	85.0461%	2144.7732	627453.20	79.6635%	0.2395	161.00	36.9701%	0.1835	172.40	29.8675%
		4	3496.1824	360878.80	76.2125%	2907.0996	1019207.20	73.1067%	0.3000	273.60	37.3526%	0.1829	183.00	27.5534%
		5	4390.4381	566911.60	66.7821%	5770.5299	1238536.80	64.9828%	0.2828	307.20	37.8477%	0.1629	181.40	26.2503%
		2	88.3146	43064.40	97.1198%	15.6384	27226.40	95.3105%	0.1421	43.20	24.1612%	0.1494	37.00	10.3782%
	D	3	257.2741	72613.60	94.2901%	42.1079	46924.80	92.2451%	0.2511	191.60	27.6250%	0.1713	95.80	12.4282%
		4	524.5129	104272.80	89.4602%	310.5687	133618.40	88.0817%	0.2546	229.80	31.8785%	0.2071	150.60	11.6104%
		5	284.3364	89650.00	82.2577%	161.1500	228266.00	81.2907%	0.2959	324.40	34.8429%	0.2178	236.40	10.3467%
		2	2833.0784	429198.20	97.5923%	273.0672	318542.40	96.0761%	0.1781	70.40	31.8593%	0.1538	50.60	21.1578%
		3	5614.4556	389241.20	95.6040%	2917.0417	1364077.60	94.0348%	0.2807	219.60	43.4234%	0.2537	200.60	31.0773%
	K	4	5881.7696	60182.00	92.3037%	6243.0403	1470989.40	91.3024%	0.3714	342.40	50.2058%	0.3039	347.20	36.0569%
		5	5833.6629	40320.20	87.7928%	7200.0440	1.00	87.1276%	0.3265	357.40	55.0982%	0.2971	383.40	39.3491%
	M	2	116.5106	50768.60	98.2674%	6.3105	9523.20	97.1800%	0.1366	55.80	25.7798%	0.1350	28.00	10.3155%
		3	174.7516	52089.60	96.3924%	50.5032	19088.40	95.1048%	0.2303	196.40	27.9106%	0.1737	71.00	8.6040%
		4	151.0538	53351.20	93.0516%	11.7353	13976.80	92.1472%	0.2293	232.80	32.3980%	0.1371	94.00	8.5263%
		5	74.8682	32522.80	87.7929%	9.0676	12475.00	87.1276%	0.2592	281.40	37.6896%	0.1382	158.00	8.9656%

TABLE 10. Average results of comparing the formulations for the 3-dimensional case (Scenario 3).

$n$	Pr.	$p$	(DOMP $N_{3I}$ )			(DOMP $N_{2I}$ )			(DOMP $N_{OT}$ )			(DOMP $N_{BEP}$ )		
			time	#nodes	%gapR	time	#nodes	%gapR	time	#nodes	%gapR	time	#nodes	%gapR
10	C	2	535.6050	315885.40	94.7568%	64.3172	98609.20	83.1911%	0.1718	101.20	26.2246%	0.1614	90.60	21.6756%
		3	3061.2795	171458.60	89.5991%	1463.1673	1489850.80	78.8776%	0.1927	132.80	25.5960%	0.1913	146.60	20.0996%
		4	3104.2662	151466.80	81.1740%	1812.8722	700664.40	73.0813%	0.2054	181.40	26.1678%	0.2605	213.80	19.8982%
		5	3058.7825	128605.20	71.3370%	2007.2649	496927.60	64.7476%	0.1783	175.40	22.7249%	0.1229	110.00	15.0102%
		2	688.6479	156903.40	98.3546%	67.9486	62430.40	94.6056%	0.2215	179.20	21.4716%	0.2629	221.80	19.0501%
	D	3	1375.4519	231147.00	96.3100%	47.7674	45659.80	92.4032%	0.2784	278.00	21.6949%	0.2494	258.80	18.7604%
		4	683.2021	149878.40	91.8585%	206.6243	98633.00	88.2517%	0.3090	354.60	21.4472%	0.3211	355.00	18.2630%
		5	267.1643	84209.60	85.2781%	187.4953	221145.20	81.6898%	0.2558	334.40	18.4730%	0.2492	338.40	15.0485%
		2	4411.1757	838744.60	98.6322%	772.6814	641139.20	95.5129%	0.2715	215.00	28.6399%	0.2633	205.20	26.6335%
		3	6083.6872	348711.00	97.1354%	5170.4426	649666.00	94.1072%	0.2733	248.20	36.8547%	0.4119	363.80	34.7234%
	K	4	6244.0137	434725.60	93.9408%	7200.1342	1.00	91.2495%	0.3939	384.20	39.6409%	0.3529	410.20	37.3418%
		5	6069.2754	142560.80	89.5868%	7200.0648	1.00	86.9273%	0.3894	493.80	39.4704%	0.3684	484.20	37.0425%
	M	2	1010.0678	173718.60	98.9966%	19.3216	23381.80	96.7080%	0.2079	183.00	26.6901%	0.2158	172.00	23.3394%
		3	2168.1779	133674.40	97.7515%	33.6661	26616.00	95.1198%	0.3011	296.00	24.8627%	0.2580	237.20	21.1599%
		4	373.2364	98523.80	94.4686%	17.3451	28016.00	92.0218%	0.3545	441.60	23.6419%	0.2867	346.00	19.0423%
		5	1567.8654	43192.40	89.6021%	21.5817	33498.60	86.9273%	0.2796	323.80	21.8016%	0.2398	332.60	16.5032%

TABLE 11. Average results of comparing the formulations for the 3-dimensional case (Scenario 4).

Sc.	$n$	$p$	Time $_{BEP}$ /StDev	Time $_{OT}$ /StDev	%NonSolved $_{BEP}$	%NonSolved $_{OT}$	%DevBest
3	20	2	9.73/7.32	11.36/12.81	0.00%	0.00%	0.00%
		5	253.35/347.87	449.32/792.30	0.00%	2.56%	0.01%
		10	46.97/191.92	77.52/305.16	0.00%	0.00%	0.00%
	30	2	59.64/42.05	148.23/216.1656875	0.00%	0.00%	0.00%
		5	2931.44/1303.68	3099.25/1082.94	75.00%	77.50%	1.63%
		10	2861.03/1394.62	3070.86/1183.70	77.50%	80.00%	3.75%
4	20	2	26.45/19.33	30.03/24.47274944	0.00%	0.00%	0.00%
		5	1865.88/1693.58	1874.01/1707.85	40.00%	40.00%	0.30%
		10	9.51/5.95	22.13/38.03	0.00%	0.00%	0.00%
	30	2	735.58/1345.89	849.51/1376.13	15.00%	15.00%	0.17%
		5	2742.49/1524.42	2836.95/1362.40	75.00%	75.00%	1.15%
		10	2745.59/1530.51	2789.27/1452.78	75.00%	75.00%	3.28%

TABLE 12. Comparison of (DOMP $N_{OT}$ ) and (DOMP $N_{BEP}$ ) for instances with  $n = 20, 30$ .

Sc.	$p$	Fr.	H0	H1	H2	BEP	t0	t1	t2	tBEP	%VarFixed	G1	G2	G0
SC1	2	M	395.3482	394.8909	395.3482	394.891	2.5	31.52	13.28	32.44	89.15%	0.00%	0.12%	0.12%
		C	19.105	18.0278	18.0278	18.0278	0.08	0.28	4.2	8.18	90.41%	0.00%	0.00%	5.64%
		K	272.8855	270.8245	270.7348	270.7348	0.84	12.88	33.24	22.92	89.84%	0.03%	0.00%	0.79%
		D	207.2299	207.2299	207.2299	207.2298	2.99	18.14	12.41	22.12	89.88%	0.00%	0.00%	0.00%
	5	M	222.3974	221.607	221.5985	222.6594	2.68	56.03	160.26	> 3600	81.54%	-0.47%	-0.48%	-0.12%
		C	18.619	18.2246	16.2491	16.2491	0.17	0.35	9.57	37.59	84.90%	10.84%	0.00%	12.73%
		K	167.6711	163.564	160.1906	160.1138	3.98	24.07	239.04	> 3600	83.05%	2.11%	0.05%	4.51%
		D	120.9466	120.0572	119.8601	119.7391	2.66	67.81	116.23	> 3600	83.02%	0.27%	0.10%	1.00%
	10	M	146.2992	144.6164	144.2134	141.8635	8.4	145.96	245.51	> 3600	85.33%	1.90%	1.63%	3.03%
		C	27.0357	22.9692	19.9944	19.9944	0.19	3.06	15.9	55.57	84.83%	12.95%	0.00%	26.04%
		K	118.7461	118.0626	117.7095	125.0503	6.34	70.23	1760.77	> 3600	85.80%	-5.92%	-6.24%	-5.31%
		D	86.6926	85.2866	85.7954	85.5946	10.84	94.08	357.67	> 3600	86.32%	-0.36%	0.23%	1.27%
SC2	2	M	399.6586	395.1866	398.2659	395.1789	2.58	96.14	27.78	462.06	77.53%	0.00%	0.78%	1.12%
		C	23.3894	21.8305	22.1818	21.7935	0.11	0.45	6.72	10.48	85.33%	0.17%	1.75%	6.82%
		K	275.9869	274.8279	274.6485	274.2601	1.4	25.67	32.68	365.95	78.30%	0.21%	0.14%	0.63%
		D	211.7095	209.7885	211.8238	209.7885	2.59	138.32	56.4	262.4	75.94%	0.00%	0.96%	0.91%
	5	M	228.6722	221.514	226.7708	223.3972	637.69	> 3600	689.41	> 3600	67.96%	-0.85%	1.49%	2.31%
		C	29.0877	23.0926	22.0907	21.5931	0.55	1.68	15.67	46.04	71.11%	6.49%	2.25%	25.77%
		K	171.2099	168.7613	167.6214	166.7242	21.55	163.7	1025.78	> 3600	64.07%	1.21%	0.54%	2.62%
		D	129.7708	125.1129	125.4487	127.2981	266.3	600.94	641.76	> 3600	69.92%	-1.75%	-1.47%	1.91%
	10	M	153.013	153.013	155.285	151.5515	3600.1	> 3600	3208.72	> 3600	76.33%	0.96%	2.40%	0.96%
		C	44.5115	35.0981	29.8835	29.842	0.66	16.66	46.47	53.42	74.23%	14.98%	0.14%	32.96%
		K	132.6658	132.6658	131.9971	142.0924	> 3600	> 3600	> 3600	> 3600	76.33%	-7.11%	-7.65%	-7.11%
		D	100.474	100.474	97.6098	108.2469	> 3600	> 3600	3215.14	> 3600	76.33%	-7.74%	-10.90%	-7.74%
SC3	2	M	404.466	394.9373	398.7388	395.7311	2.94	108.55	49.53	> 3600	62.96%	-0.20%	0.75%	2.16%
		C	28.0977	24.1051	24.1924	24.1051	0.11	0.66	6.03	10.05	74.69%	0.00%	0.36%	14.21%
		K	280.5643	278.0229	278.9015	278.0228	1.42	32.54	41.81	1455.24	52.65%	0.00%	0.32%	0.91%
		D	216.5167	210.8067	214.7534	210.6248	2.97	74.29	83.65	1488.5	72.21%	0.09%	1.92%	2.72%
	5	M	232.5595	223.4184	235.2989	240.8416	915.52	> 3600	702.09	> 3600	60.38%	-7.80%	-2.36%	-3.56%
		C	34.7835	28.4267	27.4501	26.5732	0.23	3.57	13.95	36.93	64.23%	6.52%	3.19%	23.60%
		K	183.8955	171.2615	178.0249	173.4423	128.12	1912.6	1310.85	> 3600	54.76%	-1.27%	2.57%	5.68%
		D	134.1824	133.0089	132.521	138.3565	1069.6	> 3600	1206.69	> 3600	57.96%	-4.02%	-4.40%	-3.11%
	10	M	167.5751	167.5751	171.1057	177.7685	> 3600	> 3600	3208.25	> 3600	70.41%	-6.08%	-3.89%	-6.08%
		C	50.7627	47.0082	39.3866	38.5431	1.1	4.02	25.13	32.2	70.00%	18.01%	2.14%	24.07%
		K	144.9262	144.9262	146.6831	161.303	> 3600	> 3600	2336.69	> 3600	70.41%	-11.30%	-9.97%	-11.30%
		D	110.2827	110.2827	110.5119	116.5976	> 3600	> 3600	3214.45	> 3600	70.41%	-5.73%	-5.51%	-5.73%

TABLE 13. Results of Math-Heuristic Approaches and (DOMPN<sub>BEP</sub>) in the US dataset.